2 Complex Numbers

2.1 Basic Arithmetic

1. Let $z_1 = 6 + 3i$ and $z_2 = 4 + i$. Evaluate each of the following:
   i. $z_1 + z_2$
   ii. $z_1 - z_2$
   iii. $2z_1 + 3z_2$
   iv. $z_2 - 4z_1$

2. Let $z_1 = 5 - 3i$ and $z_2 = -2 + 3i$. Evaluate each of the following:
   i. $z_2 - z_1$
   ii. $3z_1 + 2z_2$
   iii. $4z_1 - z_2$
   iv. $3z_2 - 5z_1$

2.2 Modulus and Argand Diagram

1. Evaluate each of the following:
   i. $|3 + 4i|$
   ii. $|8 - 6i|$
   iii. $|-15 + 8i|$
   iv. $|-5 - 12i|$
   v. $|2 - 2i|$
   vi. $|1 + i|$
   vii. $|-4 - 2i|$

2. $z_1 = 3 + 4i$ and $z_2 = -5 - 12i$. Plot z_1 and z_2 on an Argand diagram. Evaluate:
   i. $|z_1|$
   ii. $|z_1| + |z_2|$
   iii. $|z_1 + z_2|$
   iv. $|z_2| - |z_1|$
   v. $|z_2 - z_1|$

3. If $|-5 + ki| = 13$, find two possible values for $k, \in Z$.

4. If $|k + 15i| = 17$, find two possible values for $k, \in Z$.
5. $z_1 = 7 + 24i$ and $z_2 = 15 + ki$.
   i. Find $|z_1|$.
   ii. If $|z_1| = |z_2|$, find two possible values for $k, \in Z$.
6. If $|6 - 7i| = |2 + ki|$, find two possible values for $k, \in Z$.
7. $z = a + bi$. Prove that $|2z| = 2|z|$.
8. $z_1 = 3 + 4i$ and $z_2 = 7 + ki$. If $|z_2| = 5|z_1|$, find two possible values for $k, \in Z$.

2.3 Multiplication

1. Evaluate each of the following:
   i. $2(4 - 3i)$
   ii. $i(5 - 2i)$
   iii. $3i(2 + 4i)$
   iv. $(3 + 7i)(2 - 5i)$
   v. $(5 + 2i)(7 - i)$
   vi. $(3 + 4i)(3 - 4i)$
   vii. $-3i(1 + 3i)$
2. $z_1 = 2 + i$
   i. Find z_2 if $z_2 = iz_1$.
   ii. Plot z_1 and z_2 on an Argand diagram.
   iii. What transformation maps z_1 onto z_2?
3. $z_1 = -3 + 4i$
   i. Find z_2 if $z_2 = -iz_1$.
   ii. Plot z_1 and z_2 on an Argand diagram.
   iii. What transformation maps z_1 onto z_2
4. $z_1 = 3 - 4i$ and $z_2 = -5 + 12i$
   i. Find z_1z_2 in the form $a + bi$.
   ii. Calculate $|z_1|$, $|z_2|$ and $|z_1z_2|$.
   iii. Investigate if $|z_1| + |z_2| = |z_1 + z_2|$.
   iv. Investigate if $|z_1||z_2| = |z_1z_2|$.

2.4 Conjugate

1. $z_1 = 3 + 2i$ and $z_2 = 4 - 5i$. Calculate:
   i. $\overline{z}_1$
   ii. $\overline{z}_2$
   iii. $z_1 + \overline{z}_1$
   iv. $z_2 + \overline{z}_2$
   v. $z_1\overline{z}_1$
   vi. $z_2\overline{z}_2$

2. $z_1 = 1 + 3i$ and $z_2 = -2 - i$. Calculate:
   i. $\overline{z}_1$ and $\overline{z}_2$
   ii. $\overline{z}_1 + \overline{z}_2$
   iii. $\overline{z_1 + z_2}$
   iv. $z_1 + \overline{z}_1$
   v. $z_2\overline{z}_2$

2.5 Division

1. Write each of the following in the form $a + bi$, where $a, b \in Q$:
   i. $\frac{2+4i}{2}$
   ii. $\frac{12-9i}{3}$
   iii. $\frac{10+15i}{5}$
   iv. $\frac{-8-24i}{4}$
   v. $\frac{10+12i}{6}$

2. Write each of the following in the form $a + bi$, where $a, b \in Q$:
   i. $\frac{3+4i}{2+i}$
   ii. $\frac{-1+9i}{4+5i}$
   iii. $\frac{-16+11i}{5+2i}$
   iv. $\frac{6-8i}{4-2i}$
   v. $\frac{12-5i}{2-3i}$
   vi. $\frac{-2-8i}{5+3i}$
   vii. $\frac{3+5i}{4-2i}$

2.6 Equations with complex roots

1. Solve the equation $z^2 + 2z + 5 = 0$, giving your answer in the form $a \pm bi$, where $a, b \in Q$.

2. Solve the equation $z^2 + 6z + 13 = 0$, giving your answer in the form $a \pm bi$, where $a, b \in Q$.

3. Solve the equation $2z^2 + 3z + 4 = 0$, giving your answer in the form $a \pm bi$, where $a, b \in R$.

4. If $-2 + i$ is a root of the equation $z^2 + kz + 5$,where $k \in R$, find the value of k and write down the other root.

5. The equation $z^3 + 2z^2 - 3z - 10 = 0$ has one integer root and two complex roots. Solve the equation, expressing the non integer roots in the form $a \pm bi$, where $a, b \in Q$.

6. The equation $z^3 - z^2 + z + 39 = 0$ has one integer root and two complex roots. Solve the equation, expressing the non integer roots in the form $a \pm bi$, where $a, b \in Q$.

7. $-1 + i$ is a root of the equation $z^3 - z^2 - 4z - 6$. Find the other two roots.

8. $z = x + yi$. If $z^2 = 5 + 12i$, find two possible answers for z, giving your answer in the form $a + bi$, where $a, b \in Q$.

9. Find $\sqrt{-15 - 8i}$, giving your answer in the form $\pm(a + bi)$.

10. If $2 - 3i$ is a root of the equation $z^2 + (-3 + 2i)z + 5 - i = 0$. Find the other root in the from $a + bi$, where $a, b \in Q$.

11. If $1 + 4i$ is a solution to the equation $z^2 + k(2 + 3i)z - 5 + 14i = 0$, where $k \in R$, find the value of k and write down the other root in the form $a + bi$, where $a, b \in Z$.

12. $z = -1$ is a root of the equation $z^3 + (4 - i)z^2 + (-1 - 5i)z - 4 - 4i = 0$. Find the other two roots.

2.7 Polar Form

1. Express each of the following in polar form:
   - i. $1 + i$
   - ii. $2 - 2i$
   - iii. $1 + \sqrt{3}i$
   - iv. $-3 - \sqrt{3}i$
   - v. -6
   - vi. $8i$
   - vii. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

2. Simplify each of the following, writing your final answer in the form $r(\cos\theta + i\sin\theta)$
   - i. $\frac{-1+9i}{4+5i}$

ii. $(\sqrt{3}-i)^2$

3. Express each of the following in the form $a+bi$.

i. $3\Big(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\Big)\times 2\Big(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\Big)$

ii. $\dfrac{8\big(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\big)}{2\big(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\big)}$

2.8 De Moivre's Theorem

1. Use de Moivre's theorem to expand each of the following. Express you answer in the form $a+bi$, where $a,b\in R$.

i. $(1+i)^6$

ii. $(2-2i)^4$

iii. $(-1+\sqrt{3}i)^3$

iv. $(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)^3$

v. $(\frac{6-8i}{4-2i})^4$

vi. $(2\sqrt{3}-2i)^6$

2.9 Finding the n^{th} root

1. Write $-1+\sqrt{3}i$ in polar form. Hence use de Moivre's theorem to find $(-1+\sqrt{3}i)^{\frac{1}{2}}$, writing your answers in the $a+bi$.

2. Write $-8i$ in polar form. Hence use de Moivre's theorem to find $(-8i)^{\frac{1}{3}}$, writing your answers in the $a+bi$.

3. Write $-2-2\sqrt{3}i$ in polar form. Hence use de Moivre's theorem to find $(-2+2\sqrt{3}i)^{\frac{1}{4}}$, writing your answers in the $a+bi$.

4. Write $z=1$ in polar form. Use de Moivre's theorem to solve the equation $z^3=1$, writing your answers in the form $a+bi$. Hence show that the sum of these roots is 0

5. Use de Moivre's theorem to solve the equation $z^3=-i$, writing your answers in the $a+bi$.

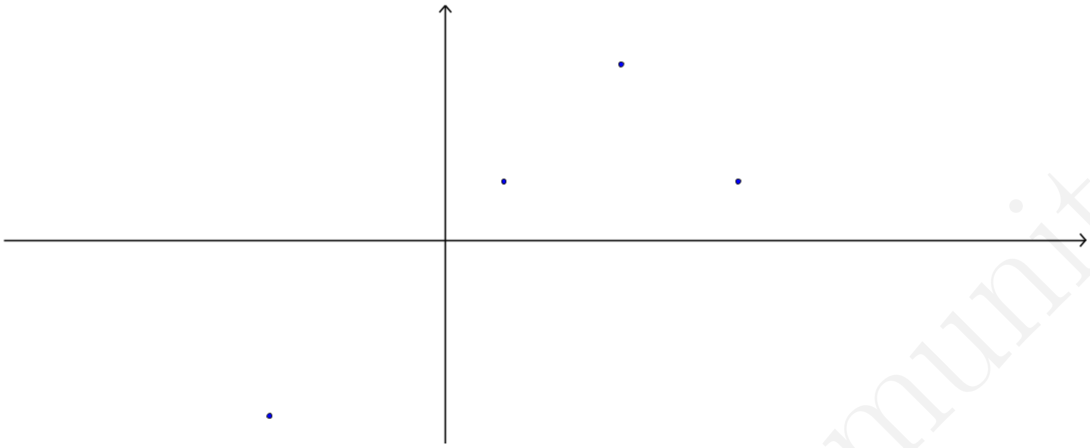
6. Use de Moivre's theorem to solve the equation $z^2=-2-2i$, writing your answers in the $a+bi$.

7. Use de Moivre's theorem to solve the equation $z^3=-27i$, writing your answers in the $a+bi$.

8. Use de Moivre's theorem to solve the equation $z^4=-4$, writing your answers in the $a+bi$.

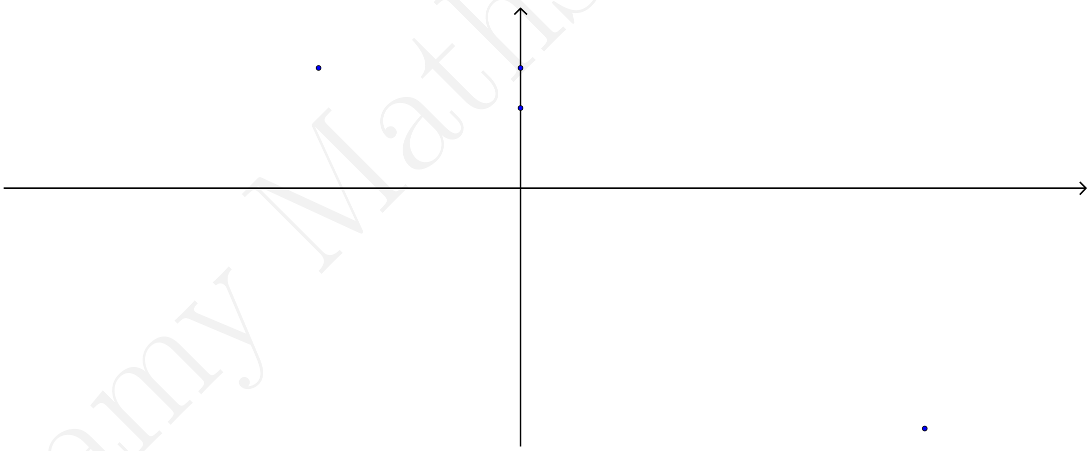
2.10 Argand Diagram

1. On the Argand diagram, label the complex numbers z_1, z_2, z_3 and z_4 based on the information given below.



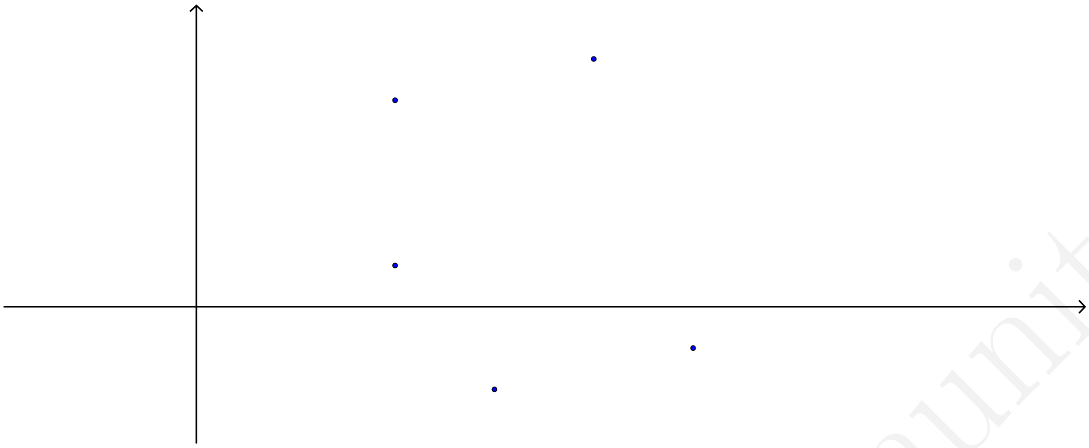
   i. $z_2 = 3z_1$
   ii. $z_3 = z_1 + 4$
   iii. $z_4 = -z_2$

2. On the Argand diagram, label the complex numbers z_1, z_2, z_3 and z_4 based on the information given below.



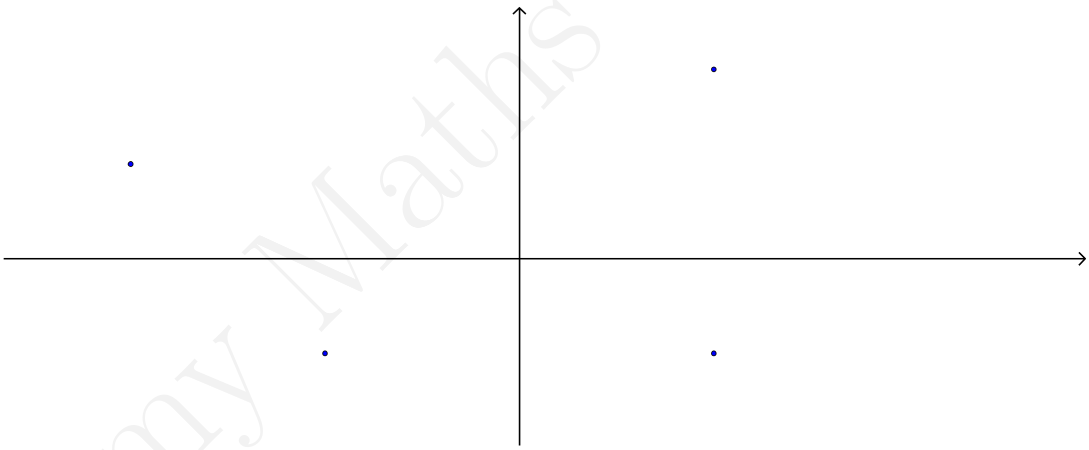
   i. $Re(z_1) = 0$
   ii. $z_2 = z_1 - i$
   iii. $z_4 = -2z_3$

3. On the Argand diagram, label the complex numbers z_1, z_2, z_3, z_4 and z_5 based on the information given below.



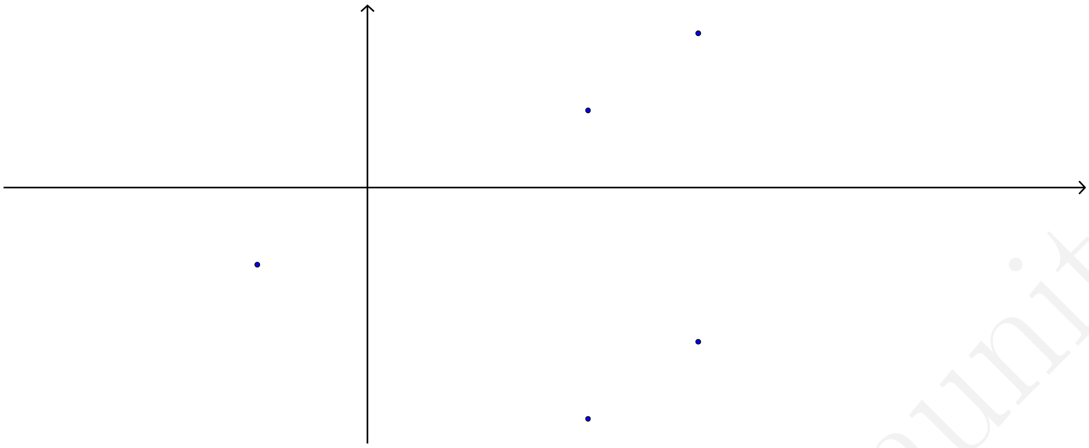
   i. $z_3 = z_1 + z_2$
   ii. $z_5 = z_4 + z_1$
   iii. $Re(z_1) = Re(z_4)$

4. On the Argand diagram, label the complex numbers z_1, z_2, z_3 and z_4 based on the information given below.



   i. $z_1 = iz_2$
   ii. $z_3 = -iz_4$
   iii. $|z_1| < |z_3|$

5. On the Argand diagram, label the complex numbers z_1, z_2, z_3, z_4 and z_5 based on the information given below.



i. $z_3 = z_1 + z_2$

ii. $|z_1| < |z_2|$

iii. $z_1 = z_3 - z_2$

iv. $z_4 = \overline{z_2}$

v. $z_5 = -iz_3$

2.11 Exam Questions

1. **2016 Paper 1**

   (a) $(-4 + 3i)$ is one root of the equation $az^2 + bz + c = 0$, where a, b, $c \in R$, and $i^2 = -1$.
   Write the other root.

   (b) Use De Moivre's Theorem to express $(1 + i)^8$ in its simplest form.

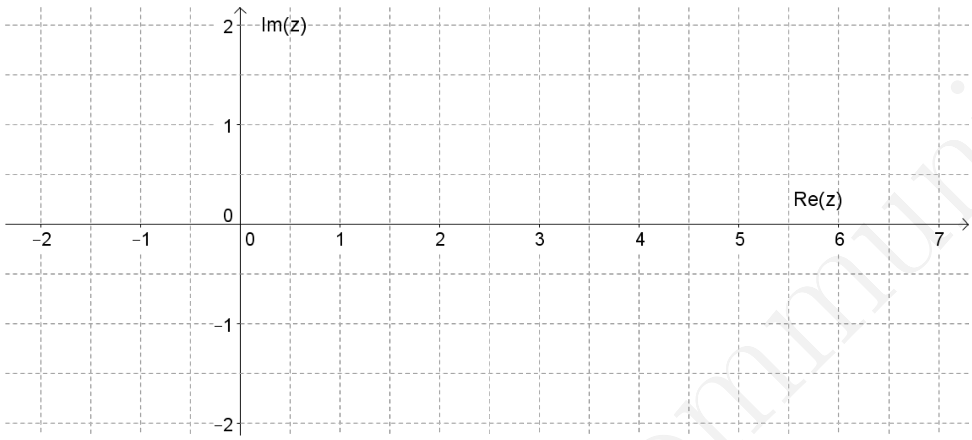
   (c) $(1 + i)$ is a root of the equation $z^2 + (-2 + i)z + 3 - i = 0$.
   Find its other root in the form $m + ni$, where m, $n \in R$, and $i^2 = -1$.

2. **2015 Paper 1**

   (a) The complex numbers z_1, z_2 and z_3 are such that $\dfrac{2}{z_1} = \dfrac{1}{z_2} + \dfrac{1}{z_3}$, $z_2 = 2 + 3i$ and $z_3 = 3 - 2i$, where $i^2 = -1$. Write z_1 in the form $a + bi$, where $a, b \in Z$.

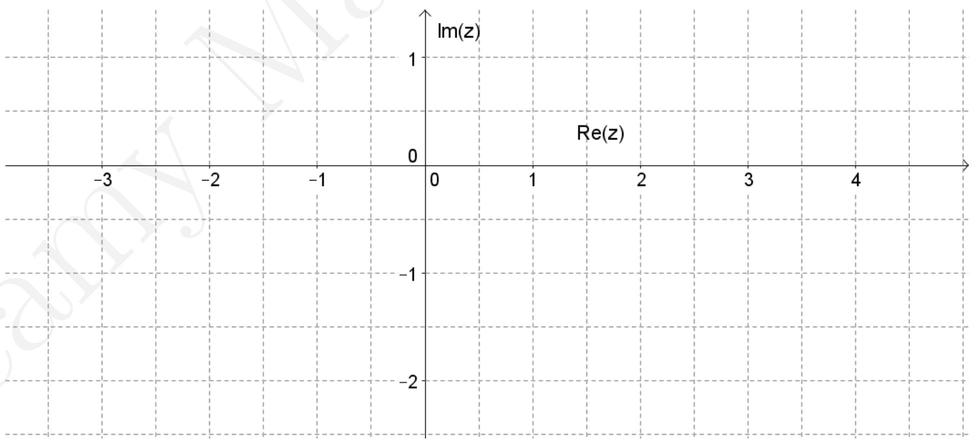
   (b) Let ω be a complex number such that $\omega^n = 1$, $\omega \neq 1$, and $S = 1 + \omega + \omega^2 + ... + \omega^{n-1}$. Use the formula for the sum of a finite geometric series to write the value of S in its simplest form.

3. **2014 Paper 1**
Let $z_1 = 1 - 2i$, where $i^2 = -1$
   (a) The complex number z_1 is a root of the equation $2z^3 - 7z^2 + 16z - 15 = 0$. Find the other two roots of the equation.
   (b) i. Let $w = z_1\bar{z_1}$, where $\bar{z_1}$ is the conjugate of z_1. Plot z_1, $\bar{z_1}$ and w on the Argand diagram and label each point.



   ii. Find the measure of the acute angle, $\bar{z_1}wz_1$, formed by joining $\bar{z_1}$ to w to z_1 on the diagram above. Give your answer correct to the nearest degree.

4. **2013 Paper 1**
$z = \frac{4}{1+\sqrt{3}i}$ is a complex number, where $i^2 = -1$.
   (a) Verify that z can be written as $1 - \sqrt{3}i$.
   (b) Plot z on an Argand diagram and write z in polar form.



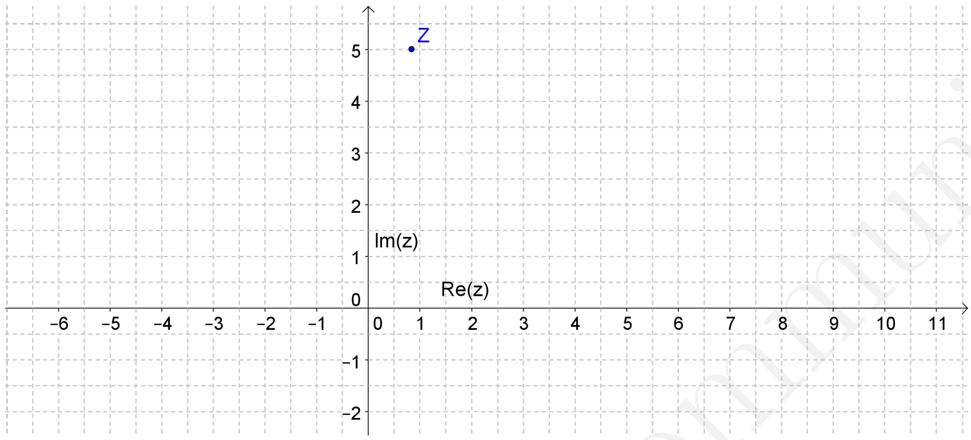
   (c) Use De Moivre's theorem to show that $z^{10} = -2^9(1 - \sqrt{3}i)$.

5. **2012 Paper 1**

The complex number z has a modulus of $5\frac{1}{16}$ and argument $\frac{4\pi}{9}$.

(a) Find, in polar form, the four complex fourth roots of z. (That is, find the four values of w for which $w^4 = z$.

(b) z is marked on the Argand Diagram below. On the same diagram, show the four answers to part **(a)**.

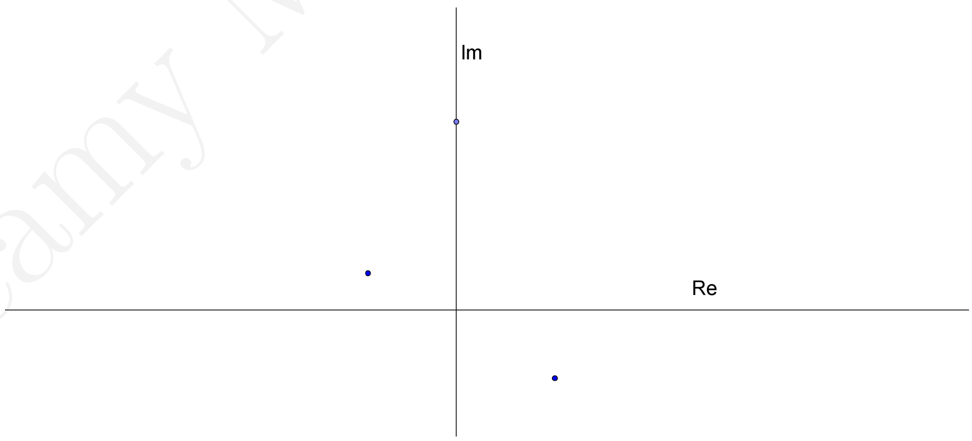


6. **2011 Paper 1**

(a) i. Write the complex number $1 - i$ in polar form.

ii. Use De Moivre's theorem to evaluate $(1 - i)^9$, giving your answer in rectangular form.

(b) A complex number z has a modulus greater than 1. The three numbers z, z_2, and z_3 are shown on the Argand diagram. One of them lies on the imaginary axis, as shown.



i. Label the points on the diagram to show which point corresponds to which number.

ii. Find θ , the argument of z.