# Course 1 - Logarithms and Exponential Functions

## 1.1 Logarithms

Solve the following equations for $x$:

1. $\log_3(2x+5)=2$
2. $\log_2(x+7)=-1$
3. $\log_2 x+\log_2(x+2)=3$
4. $\log_3(10x)-\log_3(x+1)=2$
5. $2\log_5 x-\log_5(x-1)=\log_5 4$
6. $2\log_7 x-\log_7 2=\log_7 32$
7. $\log_2(x+1)=2\log_2(x+2)-\log_2(x+5)$
8. $2\log_6(x-2)=2$
9. $\log_9 x+\log_9(x-2)=\frac{1}{2}$
10. $\log(7x-6)-2\log x=\log 2$

Solve the following pairs of equations for $x$ and $y$:

11. $\log_2(3x-2y)=2$ and $\log_3(x+2y)=\log_3 4$
12. $\log_2(x+y)=0$ and $\log_2(2x+y)=2$
13. $\log_4 x+\log_4 y=\frac{1}{2}$ and $\log_5(x+y)=\log_5 3$
14. $\log_2 4-\log_2 x=\log_2(x+y)$ and $\log_{16} 2+\log_{16}(x+y)=\frac{3}{4}$

15. Given that $\log_3 2 = a$ and $\log_3 5 = b$, express the following in terms of $a$ and $b$:

    i. $\log_3 10$
    ii. $\log_3 20$
    iii. $\log_3 \frac{5}{2}$
    iv. $\log_3 50$
    v. $\log_3 100$
    vi. $\log_3 \frac{25}{8}$
    vii. $\log_3 \frac{5}{\sqrt{2}}$
    viii. $\log_3 \frac{\sqrt{5}}{8}$
    ix. $\log_3 15$
    x. $\log_3 60$
    xi. $\log_3 \frac{6}{5}$
    xii. $\log_3 \sqrt[3]{30}$

To solve the following equations you should use the change of base formula:

16. (a) $\dfrac{\log_5(7x+1)}{3} = \log_{125}(5x+11)$
    (b) $\log_3 x = \log_9(5x-4)$
    (c) $\log_4 x + \log_2 x = \frac{3}{4}$
    (d) $\log_3(x+3) = \log_9(10x+6)$
    (e) $\log_2(x+1) + \log_8(x+1) = 4$
    (f) $\log_{25} 2 + \log_{25}(x+1) + \log_{125}(2x+2) = \frac{5}{6}$
    (g) $\log_3(x+5) + \log_2(x+5) = 4$
    (h) $\log_5(2x-1) - \log_{15}(2x-1) = 2$

17. (a) $6\log_x 2 + \log_2 x - 5 = 0$
    (b) $2\log_x 3 - \log_3 x + 1 = 0$
    (c) $\log_5 x + 2 = 3\log_x 5$
    (d) $\log_4 x + 6\log_x 4 + 5 = 0$
    (e) $\log_6 x + 2\log_x 6 = 3$

## 1.2 Introduction to Exponential Functions

1. In the year 2000, the population of the world was 6.070 Billion ; in 2015, it was 7.3 Billion.
   Use this information to create a model in the form $Y = Ae^{rt}$, where $Y$ represents the population in Billions, and $t$ is time in years.
   Assuming a model of exponential growth estimate the global population in
   (a) 2020
   (b) 2050
   (c) When will the world population reach 10 Billion?

2. A new born baby gains weight at a rate proportional to its weight during the first weeks of it's life. A baby weighing 3.6kg at birth weighs 3.72kg after one week.
   (a) Estimate its weight at three weeks.
   (b) After how many weeks will the baby weigh 5kg?

3. The value of a piece of equipment is declining exponentially according to a function of the form

$$V = V_0 e^{-rt}$$

   where $V$ equals the value of the equipment in euro and $t$ equals the age of the equipment in years. When the equipment was 3 years old, its value was €520, 000. When it was 7 years old, its value was €130, 000.
   (a) What will the equipment be worth when it is 10 years old?

4. An online dating app had 2.5 million users at the end of 2013, and 6.5 million users at the end of 2016. We want to model this growth.
   (a) Assuming the growth is exponential, create an exponential model in the form $Y = Ae^{rt}$, where $Y$ is the number of users (in millions) and $t$ is time in years.
   (b) Using this model, how many users will the dating app have at the end of 2017?
   (c) Using this model, by the end of what year do we expect the dating app to have over 20 million users?

5. China is the worlds largest country in terms of population. At the beginning of 2016 the population was estimated at 1.382 Billion people. In the year 2005 it was 1.305 Billion. India is the worlds second largest country in terms of population. At the beginning of 2016 the populatiion of India was 1.327 Billion. In the year 2005 it was 1.144 Billion.
   (a) Using exponential models, show that India's population is growing faster than China's.
   (b) At current growth rates, when will India's population overtake China's.

6. A Petri dish initially contains a sample of 500 cells. The number of cells doubles in three hours. Using a model of exponential growth.

(a) Find a formula for the number $N$ of cells present $t$ hours after the initial time.

(b) How many cells will there be after 5 hours?

(c) How long does it take the number of cells to reach 10000?

## 1.3 Half Life and Carbon Dating

7. The half-life of radium is 1690 years. What fraction of an initial amount will be present after;

   (a) 100 years?

   (b) 1000 years?

   (c) 10000 years?

8. A fossil is found to have a Carbon-14 count that is 66% of the naturally occurring amount. Assuming the half life of Carbon-14 is 5730 years what age is the fossil?

## 1.4 Newtons Law of Cooling

$$T = T_a + (T_0 - T_a)e^{-rt}$$

$T$ is the temperature of the object (variable)
$t$ is the time (variable)
$T_a$ is the ambient/surrounding temperature (constant)
$T_0$ is the initial temperature (constant)
$r$ is the cooling rate (constant)

9. A pie taken out of the oven at $220°C$ into a kitchen of temperature $25°C$ has cooled to $70°C$ after 35 minutes.

   (a) Find an equation for the temperature of the pie

   (b) Find how long it will take to reach $40°C$.

   (c) What is the temperature of the pie after 15 minutes?

10. A hot bath has a temperature $65°C$ when its drawn. After one hour the bath has cooled to $33°C$. If the ambient temperature of the bathroom is $20°C$, find the temperature of the bath;

    (a) 15 minutes after its drawn and

    (b) 30 minutes after it was drawn.

11. Surveys have shown that the best temperature to drink green tea is at $58°C$. A kettle is boiled to $100°C$ and a mug of green tea is poured. The mug cools to $85°C$ within five minutes, in a room with an ambient temperature of $22°C$. How many more minutes until the tea is at its optimum drinking temperature?

## 1.5 Exam Questions

1. **2019 Paper 1**

2. (a) Factorise fully: $3xy - 9x + 4y - 12$.

   (b) $g(x) = 3x\ln x - 9x + 4x\ln x - 12$.
   Using your answer to part (a) or otherwise, solve $g(x) = 0$.

3. **2018 Paper 1**

   The time, in days of practice, it takes Jack to learn to type $x$ words per minute (wpm) can be modelled by the function:

$$t(x) = k\left[ln\left(1 - \frac{x}{80}\right)\right], \text{ where } 0 \leq x \leq 70, x \in \mathbb{R}, \text{ and } k \text{ is a constant.}$$

   (a) Based on the function $t(x)$, Jack can learn to type 35 wpm in 35.96 days. Write the function above in terms of $k$ and **hence** show that $k = -62.5$, correct to 1 decimal place.

   (b) Find the number of wpm that Jack can learn to type with 100 days of practice. Give your answer correct to the nearest whole number.

   (c) Complete the table below, correct to the nearest whole number and hence draw the graph of $t(x)$ for $0 \leq x \leq 70$, $x \in \mathbb{R}$.

| $x$ (wpm) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|---|---|
| $t(x)$ (days) | | | | | | | | |

   (d) A simpler function that could also be used to model the number of days needed to attain $x$ wpm is $p(x) = 1.5x$.
   Draw, on your diagram from above, the graph of $p(x)$ for $0 \leq x \leq 70$, $x \in \mathbb{R}$.

   (e) Let $h(x) = p(x) - t(x)$.

   i. Use your graphs above to estimate the solution to $h(x) = 0$ for $x > 0$.

   ii. Use calculus to find the maximum value of $h(x)$ for $0 \leq x \leq 70$, $x \in \mathbb{R}$.
   Give your answer correct to the nearest whole number.

   **2017 Paper 1**

4. The amount of a substance remaining in a solution reduces exponentially over time. An experiment measures the percentage of the substance remaining in the solution. The percentage is measured at the same time each day. The data collected over the first 4 days are given in the table below. Based on the data in the table, estimate which is the first day on which the percentage of the substance in the solution will be less than 0·01%

| **Day** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **Percentage of substance (%)** | 95 | 42·75 | 19·2375 | 8·6569 |

5. Sometimes it is possible to predict the future population in a city using a function. The population in Sapphire City, over time, can be predicted using the following function:

$$p(t) = Se^{0.1t} \times 10^6$$

The population in Avalon, over time, can be predicted using the following function:

$$q(t) = 3.9e^{kt} \times 10^6$$

In the functions above $t$ is time, in years; $t = 0$ is the beginning of 2010; and both $S$ and $k$ are constants.

(a) The population in Sapphire Cite at the beginning of 2010 is 1,100,000 people. Find the value of $S$.

(b) Find the predicted population in Sapphire City at the beginning of 2015.

(c) Find the predicted change in the population in Sapphire City during 2015.

(d) The predicted population in Avalon at the beginning of 2011 is 3709795 people. Write down and solve an equation in $k$ to show that $k = -0.05$, correct to 2 decimal places.

(e) Find the year during which the populations in both cities will be equal.

(f) Find the predicted average population in Avalon from the beginning of 2010 to the beginning of 2025.

(g) Use the function $q(t) = 3.9e^{-0.05t} \times 10^6$ to find the predicted rate of change of the population in Avalon at the beginning of 2018.

**2016 Paper 1**

6. (a) i. $f(x) = \dfrac{2}{e^x}$ and $g(x) = e^x - 1$, where $x \in R$.
Complete the table below. Write your values correct to two decimal places where necessary.

| $x$ | 0 | 0.5 | 1 | ln(4) |
|---|---|---|---|---|
| $f(x) = \dfrac{2}{e^x}$ | | | | |
| $g(x) = e^x - 1$ | | | | |

ii. Use the table to draw the graphs of $f(x)$ and $g(x)$ in the domain $0 \leq x \leq \ln(4)$. Label each graph clearly.

iii. Use your graphs to estimate the value of $x$ for which $f(x) = g(x)$.

(b) Solve $f(x) = g(x)$ using algebra.

7. Given $\log_a 2 = p$ and $\log_a 3 = q$, where $a > 0$, write each of the following in terms of $p$ and $q$.

   i. $\log_a \frac{8}{3}$

   ii. $\log_a \frac{9a^2}{16}$

   **2014 Paper 1**

8. Ciaran is preparing food for his baby and must use cooled boiled water. The temperature of the water when it boils is $100°C$ and the room temperature is a constant $23°C$. The equation $y = Ae^{kt}$ describes how the boiled water cools. In this equation:

   - $t$ is the time, in minutes, from when the water boiled,
   - $y$ is the **difference** between the water temperature and the room temperature at time $t$, measured in degrees Celsius,
   - $A$ and $k$ are constants.

   (a) Write down the value of the temperature difference, $y$, when the water boils, and find the value of $A$.

   (b) After five minutes, the temperature of the water is $88°C$. Find the value of $k$, correct to three significant figures.

   (c) Ciaran prepares the food for his baby when the water has cooled to $50°C$. How long does it take, correct to the nearest minute, for the water to cool to this temperature?

   (d) Using your values for $A$ and $k$, sketch the curve $f(t) = Ae^{kt}$ for $0 \leq t \leq 100$, $t \in \mathbb{R}$

   (e) i. On the same diagram, sketch a curve $g(t) = Ae^{mt}$. showing the water cool at a **faster** rate, where $A$ is the value from part **a**, and $m$ is a constant. Label each graph clearly.

   ii. Suggest one possible value for $m$ from the sketch you have drawn and give one possible reason for your choice.

   (f) i. Find the rates of change of the function $f(t)$ after 1 minute and after 10 minutes. Give your answer correct to two decimal places.

   ii. Show that the rate of change of $f(t)$ will always increase over time.

9. **2013 Paper 1** Scientist can estimate the age of certain ancient items by measuring the proportion of carbon$-14$, relative to the carbon content in the item. The formula used is $Q = e^{-\frac{0.693t}{5730}}$, where $Q$ is the proportion of carbon$-14$ remaining and$t$ is the age, in years, of the item.

   (a) An item is 2000 years old. Use the formula to find the proportion of carbon$-14$ in the item.

   (b) The proportion of carbon$-14$ in an item found at Lough Boora, County Offaly, was 0.3402. Estimate, correct to two significant figures, the age of the item.

10. **2011 Paper 1** In a science experiment, a quantity $Q(t)$ was observed at various points in time. $Q$ follows a rule of the form $Q(t) = Ae^{bt}$, where $A$ and $b$ are constants. Time is measured in seconds from the instant of the first observation. The table below gives the results.

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $Q(t)$ | 2.920 | 2.642 | 2.391 | 2.163 | 1.957 |

(a) Use any two of the observations from the table to find a value for $A$ and the value for $b$, correct to three decimal places.

(b) Use a different observation from the table to verify your values for $A$ and $b$.

(c) Show that $Q(t)$ is a constant multiple of $Q(t-1)$, for $t \geq 1$

(d) Find the value of the constant $k$ for which $Q(t+k) = \frac{1}{2}Q(t)$, for all $t \geq 0$. Give your answer correct to two decimal places.