3.8 Slopes of Tangents

1. Find the equation of the tangent to the curve at the indicated point in each of the following functions:
   (a) $y = x^2 - 5x + 4$ at $(2, -2)$
   (b) $y = 2x + \frac{2}{x}$ at $(\frac{1}{2}, 5)$
   (c) Find the equation of the tangent to the curve $y = 3x^2 - 2x + 1$ at the point $(1, 2)$.
2. Find the equation of the tangent to the curve $y = \frac{3x^2}{1-2x^2}$ at the point where $x = 1$.
3. Find the equation of the tangent to the curve $y = 2e^{5x}$ at the point where $x = 0$.
4. At what point does the function $y = 3x^2 - 4x + 7$ have a slope of 2?
5. Find the coordinates where the tangent to the function $y = 2x^3 - 3x^2 - 13x + 4$ is parallel to the line $x + y - 5 = 0$.
6. Find the coordinates where the tangent to the function $y = x^3 - 3x^2 - 3x - 2$ has a slope of 21.
7. Given that the function $y = ax^2 + bx + 2$ has slope -1 at the point $(-1, 0)$, where a and b are constants. Find the values of a and b.
8. Given that the function $y = px^2 + qx + 1$ has slope 9 at the point $(3, 10)$, where p and q are constants. Find the values of p and q.
9. The curve $y = \frac{ax+2}{bx-1}$ has a slope -7 at the point $(1, 5)$. Find the values of the constants a and b, where $a > 0$ and $b > 0$.
10. A function is given as $y = ax^3 + bx^2 + cx + d$ where a, b, c and d are constants. The function contains the points $(0, -6)$ and $(-2, 0)$. At these points the slopes are -7 and 5, respectively.
    (a) Find the values of a, b, c and d.
    (b) Find the coordinates of the points where the curve cuts the x-axis.
    (c) Hence sketch the curve.
11. The equation of a curve is $y = ax^3 + bx^2 + cx + d$ where a, b, c and d are constants. The curve has a slope of -4 at the point $(1, -20)$ and the slope is -7 at $(-2, 10)$.
    (a) Find the values of a, b, c and d.
    (b) Show that the curve crosses the x-axis at the point $(-4, 0)$.
    (c) Find the coordinates of the other two points where the curve cuts the x-axis.
    (d) Hence sketch the curve.
12. Show that $f(x) = x^3 - 3x^2 + 12x - 7$ is an increasing function for all values $x \in R$.
13. Show that the curve $y = \frac{3x+5}{2x+1}$ is decreasing for all $x \neq -2$.

3.9 Maximum and Minimum Turning Points

1. Find the turning point of the function $f(x) = 2x^2 - 8x + 7$ and show that it is a local minimum.

2. Find and classify the coordinates of the turning points of each of the following curves;
   (a) $f(x) = 2x^3 - 9x^2 + 12x - 3$
   (b) $y = x^2e^{2x}$
   (c) $y = \frac{3x^2}{x+3}$

3. Find and classify the coordinates of the turning points of the function $f(x) = 5x + \frac{20}{x}$

4. A function is defined by $f(x) = 2\ln(x) - 9x^2, x > 0$
   (a) What is the slope of the tangent when $x = 1$?
   (b) Find and classify the coordinates of the turning point of the function.

5. Find and classify the turning point of the function $y = \ln(5x^2) + 2x$.

6. Find the turning points of the curve $y = (x^2 - 10x + 25)e^{\frac{2x}{5}}$ and state which point is the local max and local min.

7. A function is defined by $f(x) = -xe^{-kx}$,$x \in R$ k a constant and $k > 0$. Find and classify the turning point in terms of k.

8. A function is defined as $f(x) = \sin(3x)e^{-3x}$, where $0 \leq x \leq \pi$. Find and classify the turning point of the function.

9. The function $y = ax^3 + bx^2 + cx + d$ has a turning point at $(-6, 256)$ and $(2, 0)$.

   Find the values of a, b and c and d

10. Find the coordinates of the point of inflection of each of the following curves:
    (a) $y = 2x^3 - 3x^2 - 12x + 30$
    (b) $y = 15 - 12x + 9x^2 - x^3$

11. The function $f(x) = ax^3 + bx^2 + cx + d$ has a minimum point at $(2, -8)$ and a point of inflection at $(-1, 46)$.

    Find the values of a, b, c and d.

12. The function $f(x) = e^x(ax^2 + bx)$ has a turning point at $(-3, \frac{9}{e^3})$. Find the value of a and the value of b.

13. A function is defined by $y = x^3 - 6kx^2 + 32$, where k is a constant.
    (a) Show that one of the turning points is independent of k.
    (b) Find the value of k for which the function has only one real root.

3.10 Related rates of change

1. The radius of a circle is increasing at a rate of 5 cm/s. Find the rate of change of the area, when the radius is 6 cm.

2. The area of a circle is increasing at a rates of 56π cm^2/s. Find the rate of increase of the radius, when the radius is 7 cm.

3. The side of a square, with side length x, is increasing at a rate of 10 cm/s. Find the rate of change of the area, when the side length is 3 cm.

4. The area of a square, with side length x, is increasing at a rate of 100 cm^2/s. Find the rate of increase of the side, when the area is 25 cm^2

5. The radius of a sphere is increasing at a rate of 6 cm/s.
   i. Find the rate of increase of the volume, when the radius is 4 cm.
   ii. Find the rate of increase of the surface area, when the radius is 5 cm.

6. The volume of a sphere is increasing at a rate of 48π cm^3/s.
   i. Find the rate of change of the radius in terms of r.
   ii. Find the rate of change of the radius when r=3 cm.
   iii. Find the rate of change of the surface area, when r=4 cm.

7. A cylinder is such that its height is the same as its diameter.
   i. Write down the formula for the volume of this cylinder, in terms of r.
   ii. The radius of this cylinder is increasing at a rate of 1 m/s. Find the rate of increase of the volume of the cylinder, when the radius is 50 cm.

8. A stone is dropped into a liquid, creating a circular ripple, with a radius expanding at a rate of 2 cm/s. Find the rate of increase of the area of this circular ripple, after 5 seconds.

9. A cone has a height that is five times the length of the radius.
   (a) Write an expression for the volume in terms of h.
   (b) A cone has radius 1 cm and height 5 cm. Water is being poured into this cone at a rate of 3 cm^3/s.
      i. Find the rate at which the water is rising ($\frac{dh}{dt}$), when the cone is half full.
      ii. Find the rate at which the area of the free surface (circle) is increasing, when the cone is a quarter full.

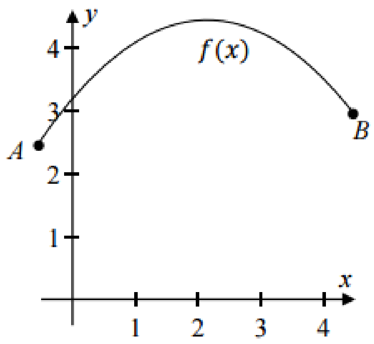
3.11 Exam Questions

1. **(2016 Paper 1)**
   i. Show that $f(x) = 3x - 2$, where $x \in R$, is an injective function.
   ii. Given that $f(x) = 3x - 2$, where $x \in R$, find a formula for f^{-1}, the inverse function of f. Show your work.

2. (a) Differentiate the function $(2x + 4)^2$ from first principles, with respect to x.
   (b) i. If $y = x \sin\left(\frac{1}{x}\right)$, find $\frac{dy}{dx}$.
   ii. Find the slope of the tangent to the curve $y = x \sin\left(\frac{1}{x}\right)$, when $x = \frac{4}{\pi}$.
   Give your answer correct to two decimal places.

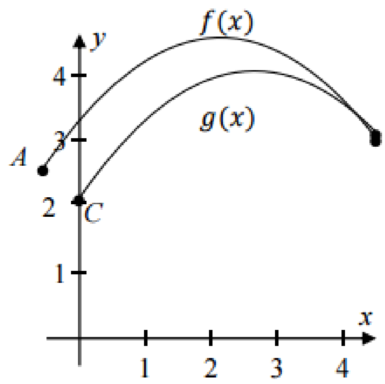
3. (a) i. Air is pumped into a spherical exercise ball at the rate of 250 cm^3 per second. Find the rate at which the radius is increasing when the radius of the ball is 20 cm.
   Give your answer in terms of π.
   ii. Find the rate at which the surface area of the ball is increasing when the radius of the ball is 20 cm.

4. (a) The diagram shows Sarah's first throw at the basket in a basketball game. The ball left her hands at A and entered the basket at B. Using the co-ordinate place with $A(-0.5, 2.565)$ and $B(4.5, 3.05)$, the equation of the path of the centre of the ball is
$$f(x) = -0.274x^2 + 1.193x + 3.23,$$
where both x and $f(x)$ are measured in metres.



   i. Find the maximum height reached by the centre of the ball, correct to three decimal places.
   ii. Find the acute angle to the horizontal at which the ball entered the basket. Give your angle correct to the nearest degree.

   iii. Sarah took a second throw. This throw followed the path of the parabola $g(x)$ as shown.
   The ball left Sarah's hand at teh point $C(0, 2)$.
   The graph $y = g(x)$ is the image of the graph $y = f(x)$ under the translation which maps A to C.
   Using your result from part **a(i)**, show that the centre of this ball reached its maximum height at the point $(2.677, 3.964)$, correct to three decimal places.



   iv. Hence, or otherwise, find the equation of the parabola $g(x)$.

(b) The heptathlon is an Olympic competition. It consists of seven events including the 200m race and the javelin. The scoring system uses formulas to calculate a score for each event. The table below shows the formulas for two of the events and the values of constants used in these formulas, where x is the time taken (in seconds) or distance achieved (in metres) by the competitor and y is the number of points scored in the event.

Event	x	**Formula**	a	b	c
200 m race	Time (s)	$y = a(b - x)^c$	4·99087	42·5	1·81
Javelin	Distance (m)	$y = a(x - b)^c$	15·9803	3·8	1·04

i. In the heptathlon, Jessica ran 200m in 23.8 s and threw the javelin 52.8 m. Use the formulas in the table to find the number of points she scored in each of these events, correct to the nearest point.

ii. The world record distance for the javelin in the heptathlon, would merit a score of 1295 points. Find the world record distance for the javelin, in the heptathlon, correct to two decimal places.

iii. The formula used to calculate the points for the 800m race, in the heptathlon, is the same formula used for the 200 m race but with different constants.
Jessica ran the 800 m race in 2 minutes and 1.84 seconds which merited 1087 points.
If $a = 0.11193$ and $b = 254$ for the 800 m race, find the value of c for this event, correct to two decimal places.

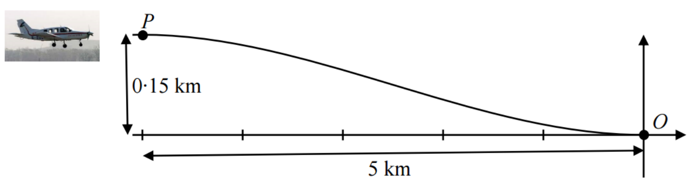
5. **(2015 Paper 1)**

(a) Solve the equation $x = \sqrt{x+6}$, $x \in R$.

(b) Differentiate $x - \sqrt{x+6}$ with respect to x.

(c) Find the co-ordinates of the turning point of the function $y = x - \sqrt{x+6}$, $x \geq -6$.

6. **(2015 Paper 1)**
A plane is flying horizontally at P at a height of 150 m above level ground when it begins its descent. P is 5 km, horizontally, from the point of touchdown O. The plane lands horizontally at O.



Taking O as the origin, $(x, f(x))$ approximately describes the path of the plane's descent where $f(x) = 0.0024x^3 + 0.018x^2 + cx + d$, $-5 \leq x \leq 0$, and both x and $f(x)$ are measured in km.

(a) i. Show that $d = 0$.

ii. Using the fact that P is the point $(-5, 0.15)$, or otherwise, show that $c = 0$.

(b) i. Find the value of $f'(x)$, the derivative of $f(x)$, when $x = -4$.

ii. Use your answer to part **(b) (i)** above to find the angle at which the plane is descending when it is 4 km from touchdown. Give your answer correct to the nearest degree.

(c) Show that $(-2.5, 0.075)$ is the point of inflection of the curve $y = f(x)$.

(d) i. If (x, y) is a point on the curve $y = f(x)$, verify that $(-x-5, -y+0.15)$ is also a point on $y = f(x)$.

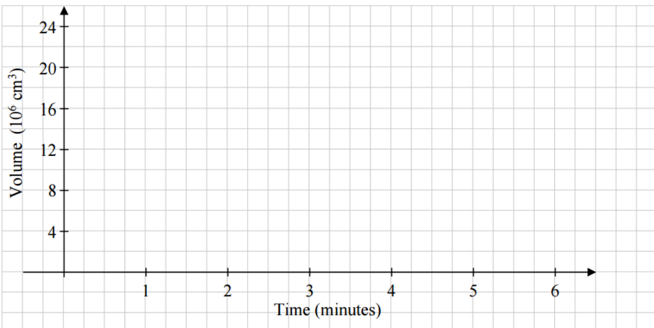
ii. Find the image of $(-x-5, -y+0.15)$ under symmetry in the point of inflection.

7. **(2015 Paper 1)**
An oil-spill occurs off-shore in an area of calm water with no currents. The oil is spilling at a rate of 4×10^6 cm^3 per minute. The oil floats on top of the water.

(a) i. Complete the table below to show the total volume of oil on the water after each of the first 6 minutes of the oil-spill.

Time (minutes)	1	2	3	4	5	6
Volume (10^6cm^3)		8				

ii. Draw a graph to show the total volume of oil on the water over the first minutes.



iii. Write an equation for $V(t)$, the volume of oil on the water, in cm^3, after t minutes.

(b) The spilled oil forms a circular oil slick **1 millimetre** thick.

i. Write an equation for the volume of oil in the slick, in cm^3, when the radius is r cm.

ii. Find the rate, in cm per minute, at which the radius of the oil slick is increasing when the radius is 50 m.

(c) Show that the area of water covered by the oil slick is increasing at a constant rate of 4×10^7 cm^2 per minute.

(d) The nearest land is 1 km from the point at which the oil spill began. Find how long it will take for the oil slick to reach land. Give your answer correct to the nearest hour.

8. **(2015 Paper 1)**
The approximate length of the day in Galway, measured in hours from sunrise to sunset, may be calculated using the function

$$f(t) = 12.25 + 4.75 \sin\left(\frac{2\pi}{365}t\right),$$

where t is the number of days after March 21$^{\text{st}}$ and $\left(\frac{2\pi}{365}t\right)$ is expressed in radians.

(a) Find the length of the day in Galway on June 5$^{\text{th}}$ (76 days after March 21$^{\text{st}}$). Give your answer in hours and minutes, correct to the nearest minute.

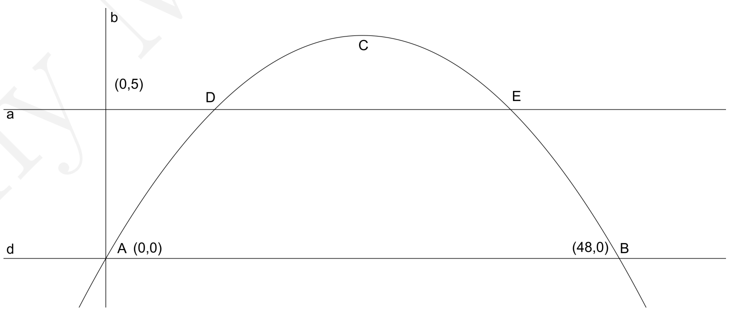
(b) Find a date on which the length of the in Galway is approximately 15 hours.

(c) Find $f'(t)$, the derivative of $f(t)$.

(d) Hence, or otherwise, find the length of the longest day in Galway.

(e) Use integration to find the average length of the day in Galway over the six months form March 21$^{\text{st}}$ to September 21$^{\text{st}}$ (184 days). Give you answer in hours and minutes, correct to the nearest minute.

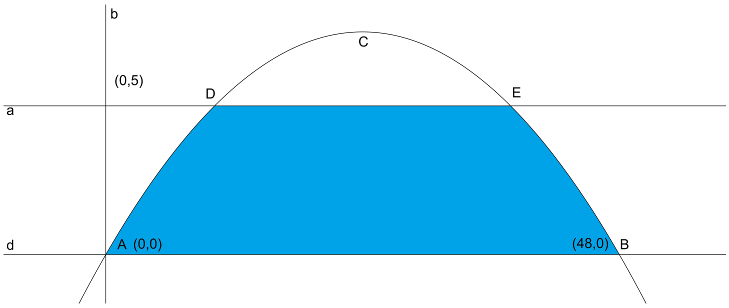
9. **(2014 Paper 1)**
In 2011, a new footbridge was opened at Mizen head, the most south-westerly point of Ireland. The are of the bridge is in the shape of a parabola, as shown. The length of the span of arch, [AB], is 48 metres.



(a) Using the co-ordinate plane, with $A(0,0)$ and $B(48,0)$, the equation of the parabola is $y = -0.013x^2 + 0.624x$: Find the co-ordinates of C, the highest point of the arch.

(b) The perpendicular distance between the walking deck, $[DE]$, and $[AB]$ is 5 metres. Find the co-ordinates of D and of E. Give your answer correct to the nearest whole number.

(c) Using integration, find the area of the shaded region, $ABED$, shown in the diagram below. Give your answer correct to the nearest whole number.



(d) Write the equation of the parabola in part **(a)** in the form $y - k = p(x - h)^2$, where k, p and h are constants.

(e) Using what you learned in part **(d)** above, or otherwise, write down the equation of a parabola for which the coefficient of x^2 is -2 and the co-ordinates of the maximum point are $(3, -4)$.

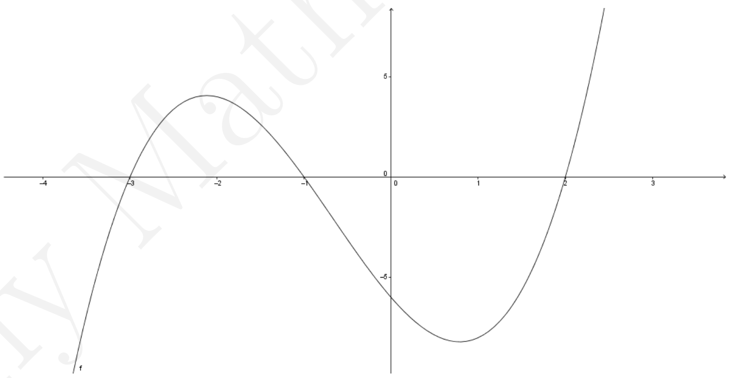
10. **2014 Paper 1**

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from the first principles.

(b) Let $f(x) = \frac{2x}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$. Find the co-ordinates of the points at which the slope of the tangent to the curve $y = f(x)$ is $\frac{1}{4}$.

11. **2014 Paper 1**

(a) The graph of a cubic function $f(x)$ cuts the $x - axis$ at $x = -3$, $x = -1$ and $x = 2$ and the $y - axis$ at $(0, -6)$, as shown. Verify that $f(x) = x^3 + 2x^2 - 5x - 6$



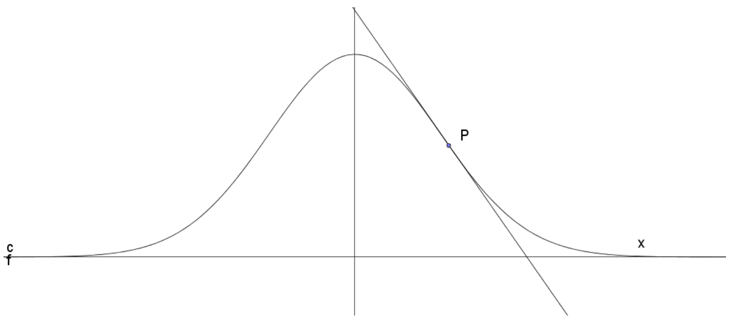
(b) i. The graph of the function $g(x) = -2x - 6$ intersects the graph of the function $f(x)$ above. Let $f(x) = g(x)$ and solve the resulting equation to find the co-ordinates of the points where the graphs of $f(x)$ and $g(x)$ intersect.

ii. Draw the graph of the function $g(x) = -2x - 6$ on the diagram above

12. **2012 Paper 1**

(a) Let $f(x) = e^{-\frac{1}{2}x^2}$ Show that the second derivative of $f(x)$ with respect to x is $f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$

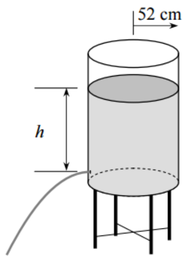
(b) The point P in the first quadrant is a point of inflection of the curve $y = e^{-\frac{1}{2}x^2}$. Show that the tangent at P crosses the $x - axis$ at $(2, 0)$.



13. **2014 Sample Paper 1**
A cubic function f is defined for $x \in \mathbb{R}$ as $f(x) : x^3 + (1 - k^2)x + k$, where k is a constant.
(a) Show that $-k$ is a root of f.
(b) Find, in terms of k, the other two roots of f.
(c) Find the set of values of k for which f has exactly one real root

14. **2014 Sample Paper 1** A is the closed interval $[0, 5]$. That is, $A = x|0 \leq x \leq 5, x \in \mathbb{R}$. The function f is defined on A by: $f(A) \to \mathbb{R} : x \mapsto x^3 - 5x^2 + 3x + 5$
(a) Find the maximum and minimum values of f
(b) State whether f is injective. Give a reason for your answer.

15. **2012 Paper 1**
An open cylindrical tank of water has a hole near the bottom. The radius of the tank is 52 cm. The hole is a circle of radius 1 cm. The water level gradually drops as water escapes through the hole.



Over a certain 20 minute period, the height of the surface of the water is given by the formula:

$$h = \left(10 - \frac{t}{200}\right)^2$$

where h is the height of the surface of the water, in cm, as measured from the centre of the hole,
and t is the time in seconds from a particular instant $t = 0$.
(a) What is the height of the surface at time $t = 0$?

(b) After how many seconds will the height of the surface be 64 cm?

(c) Find the rate at which the **volume** of the water in the tank is decreasing at the instant when the height is 64 cm. Give your answer correct to the nearest cm^3 per second.

(d) The rate at which the volume of water in the tank is decreasing is equal to the speed of the water coming out of the hole, multiplied by the area of hole. Find the speed at which the water is coming out of the hole at the instant when the height is 64 cm.

(e) Show that, as t varies, the speed of the water coming out of the hole is a constant multiple of $\sqrt{h}$.

(f) The speed, in centimetres per second, of water coming out of a hole like this is known to be given by the formula

$$v = c\sqrt{1962h}$$

where c is a constant that depends on certain features of the hole. Find, correct to one decimal place, the value of c for this hole.