Complex Numbers
Revision Series 2016

Stephen King

March 19 , 2017

Introduction

A complex number is a number expressed in the form $a + bi$, where

$$i = \sqrt{-1}, \text{ so} \quad i^2 = -1$$

In the expression $z = a + bi$, a is the coefficient of the real part, and b is the coeffiecient of the imaginary part.

Adding and Subtracting Complex Numbers

To add Complex numbers, simply add the Reals to Reals, and Imaginaries to Imaginaries.

$$
\begin{aligned}
z_1 &= 3 + 2i \\
z_2 &= 4 - 5i \\
z_1 + z_2 &=
\end{aligned}
$$

Adding and Subtracting Complex Numbers

To add Complex numbers, simply add the Reals to Reals, and Imaginaries to Imaginaries.

$$z_1 = 3 + 2i$$
$$z_2 = 4 - 5i$$
$$z_1 + z_2 = 7 - 3i$$

$z_1 = 3 + 2i \quad z_2 = 4 - 5i$

$$z_1 - z_2 =$$

$z_1 = 3 + 2i \quad z_2 = 4 - 5i$

$$z_1 - z_2 = 3 + 2i - (4 - 5i)$$

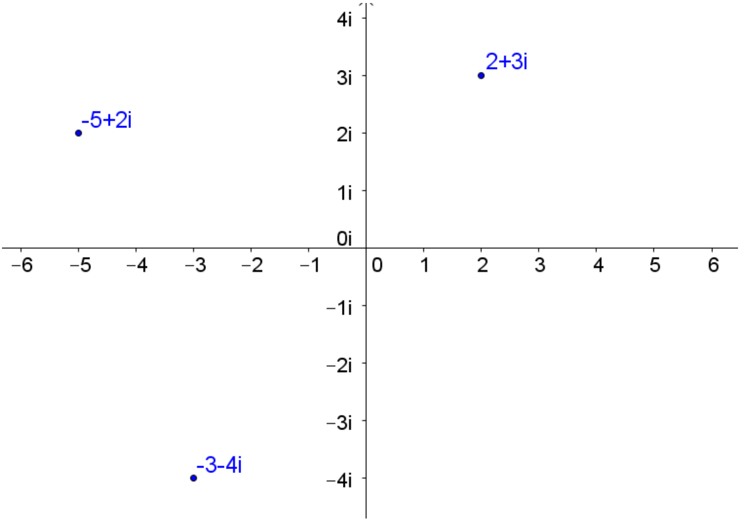
$z_1 = 3 + 2i \qquad z_2 = 4 - 5i$

$$\begin{aligned} z_1 - z_2 &= 3 + 2i - (4 - 5i) \\ &= 3 + 2i - 4 + 5i \end{aligned}$$

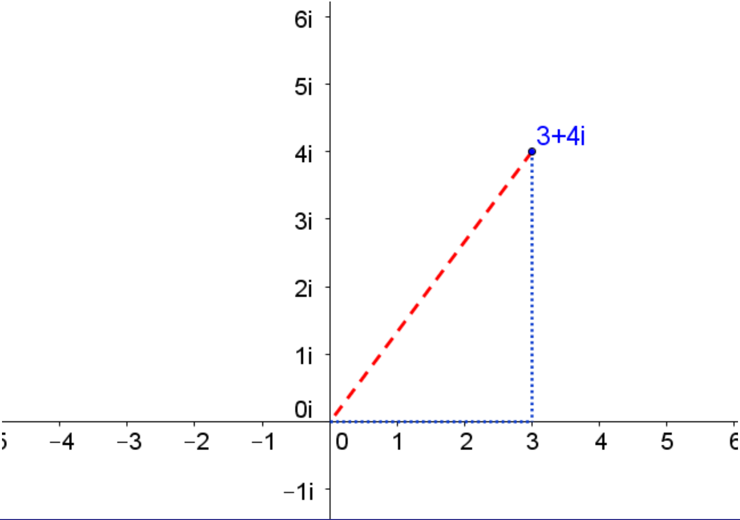
$z_1 = 3 + 2i \qquad z_2 = 4 - 5i$

$$
\begin{aligned}
z_1 - z_2 &= 3 + 2i - (4 - 5i) \\
&= 3 + 2i - 4 + 5i \\
&= -1 + 7i
\end{aligned}
$$

Argand Diagram



Modulus



Modulus

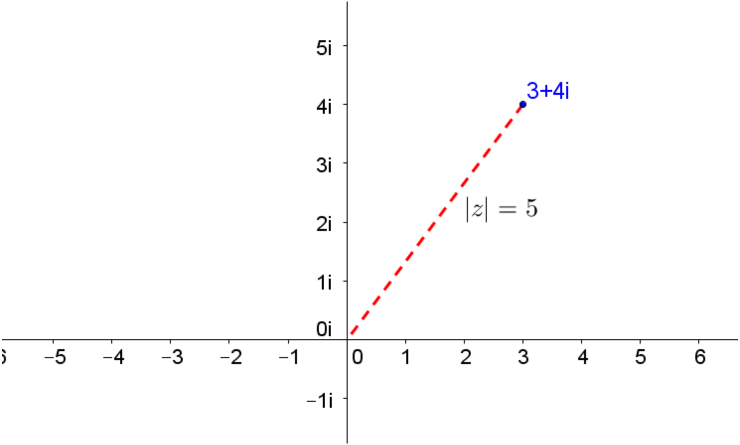
If $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}$$

$z = 3 + 4i$

$$|z| = \sqrt{(3)^2 + (4)^2} \quad = \sqrt{25} = 5$$

Modulus



$z_1 = 3 + 2i \quad z_2 = 4 - 5i$

When multiplying complex numbers remember that $i^2 = -1$

$z_1 = 3 + 2i \quad z_2 = 4 - 5i$

When multiplying complex numbers remember that $i^2 = -1$

$$z_1 z_2 = (3 + 2i)(4 - 5i)$$

$z_1 = 3 + 2i \quad z_2 = 4 - 5i$

When multiplying complex numbers remember that $i^2 = -1$

$$\begin{aligned} z_1 z_2 &= (3+2i)(4-5i) \\ &= 12 - 15i + 8i - 10i^2 \end{aligned}$$

$z_1 = 3 + 2i \quad z_2 = 4 - 5i$

When multiplying complex numbers remember that $i^2 = -1$

$$
\begin{aligned}
z_1 z_2 &= (3 + 2i)(4 - 5i) \\
&= 12 - 15i + 8i - 10i^2 \\
&= 12 - 15i + 8i - 10(-1)
\end{aligned}
$$

$z_1 = 3 + 2i \quad z_2 = 4 - 5i$

When multiplying complex numbers remember that $i^2 = -1$

$$\begin{aligned} z_1 z_2 &= (3 + 2i)(4 - 5i) \\ &= 12 - 15i + 8i - 10i^2 \\ &= 12 - 15i + 8i - 10(-1) \\ &= 22 - 7i \end{aligned}$$

Complex Conjugate

If $z = a + bi$ then $\bar{z} = a - bi$

Complex Conjugate

If $z = a + bi$ then $\bar{z} = a - bi$

$$z.\bar{z} = (a + bi)(a - bi)$$

Complex Conjugate

If $z = a + bi$ then $\bar{z} = a - bi$

$$\begin{aligned} z.\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2 i^2 \end{aligned}$$

Complex Conjugate

If $z = a + bi$ then $\bar{z} = a - bi$

$$\begin{aligned} z.\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 - b^2(-1) \end{aligned}$$

Complex Conjugate

If $z = a + bi$ then $\bar{z} = a - bi$

$$\begin{aligned} z.\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2 \end{aligned}$$

Complex Conjugate

If $z = a + bi$ then $\bar{z} = a - bi$

$$\begin{aligned} z.\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2 \end{aligned}$$

When a complex number is multiplied by it's conjugate, the product is a real number.

Dividing Complex Numbers

Simplify $\dfrac{3+2i}{4-3i}$ in the form $a+bi$

Dividing Complex Numbers

Simplify $\dfrac{3+2i}{4-3i}$ in the form $a+bi$

$$\frac{3+2i}{4-3i} \times \frac{(4+3i)}{(4+3i)} = \frac{12+9i+8i+6i^2}{16-9i^2}$$

Dividing Complex Numbers

Simplify $\dfrac{3+2i}{4-3i}$ in the form $a+bi$

$$\frac{3+2i}{4-3i} \times \frac{(4+3i)}{(4+3i)} = \frac{12+9i+8i+6i^2}{16-9i^2}$$

$$= \frac{6+17i}{25}$$

Dividing Complex Numbers

Simplify $\dfrac{3+2i}{4-3i}$ in the form $a+bi$

$$\begin{aligned}\frac{3+2i}{4-3i}\times\frac{(4+3i)}{(4+3i)} &= \frac{12+9i+8i+6i^2}{16-9i^2}\\ &= \frac{6+17i}{25}\\ &= \frac{6}{25}+\frac{17}{25}i\end{aligned}$$

Equations with Complex Roots

Solve the equation $z^2 - 4z + 13 = 0$

Equations with Complex Roots

Solve the equation $z^2 - 4z + 13 = 0$

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

Equations with Complex Roots

Solve the equation $z^2 - 4z + 13 = 0$

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$z = \frac{4 \pm \sqrt{16 - 52}}{2}$$

Equations with Complex Roots

Solve the equation $z^2 - 4z + 13 = 0$

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$z = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$z = \frac{4 \pm \sqrt{-36}}{2}$$

Equations with Complex Roots

$$z = \frac{4 \pm \sqrt{-36}}{2}$$

Equations with Complex Roots

$$z = \frac{4 \pm \sqrt{-36}}{2}$$

$$\sqrt{-36} = \sqrt{36}\sqrt{-1}$$

Equations with Complex Roots

$$z = \frac{4 \pm \sqrt{-36}}{2}$$

$$\sqrt{-36} = \sqrt{36}\sqrt{-1}$$
$$\sqrt{-36} = 6i$$

Equations with Complex Roots

$$z = \frac{4 \pm \sqrt{-36}}{2}$$

$$\sqrt{-36} = \sqrt{36}\sqrt{-1}$$
$$\sqrt{-36} = 6i$$

$$z = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

Equations with Complex Roots

Solve the equation $z^2 - 4z + 13 = 0$

$$z = 2 + 3i \text{ and } z = 2 - 3i$$

The coefficients of the equation were real, but the solutions were complex. The solutions were also a conjugate pair.

Equations with Complex Roots

Solve the equation $z^2 - 4z + 13 = 0$

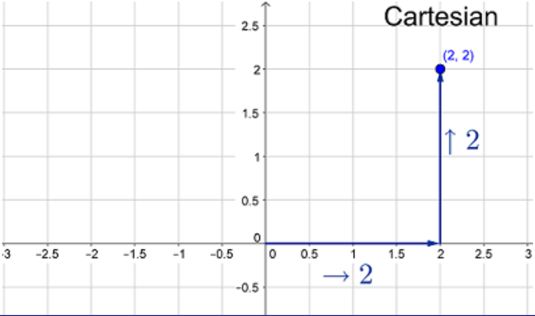
$$z = 2 + 3i \text{ and } z = 2 - 3i$$

The coefficients of the equation were real, but the solutions were complex. The solutions were also a conjugate pair.

For any quadratic equation that has real coefficients and complex roots, the roots will always be a conjugate pair!

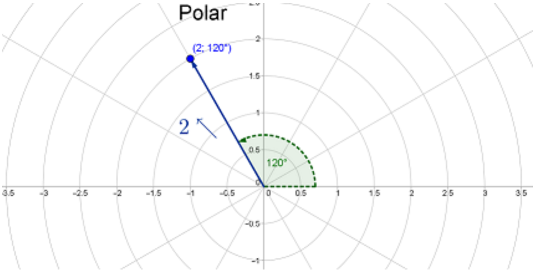
Polar Form

Polar form is another way to represent the position of a point on the plane, rather than describing the position of a point as so many units left or right of the origin (x coordinate), then as so many units above or below the origin (y coordinate)



Polar Form

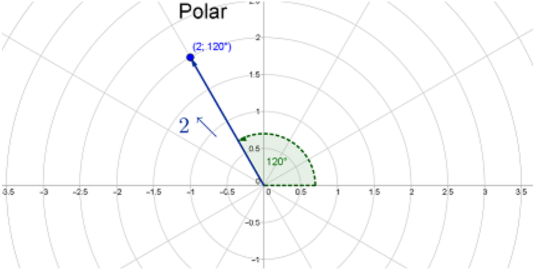
In Polar form we describe the position of a point using two measurements:



Polar Form

In Polar form we describe the position of a point using two measurements:

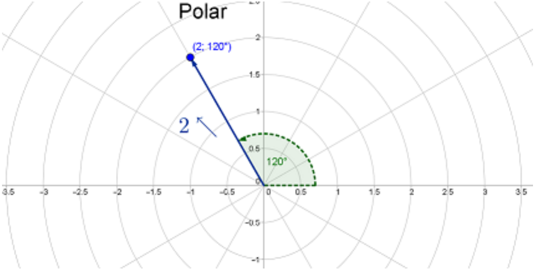
- The straight line distance of the point from the origin (the modulus).



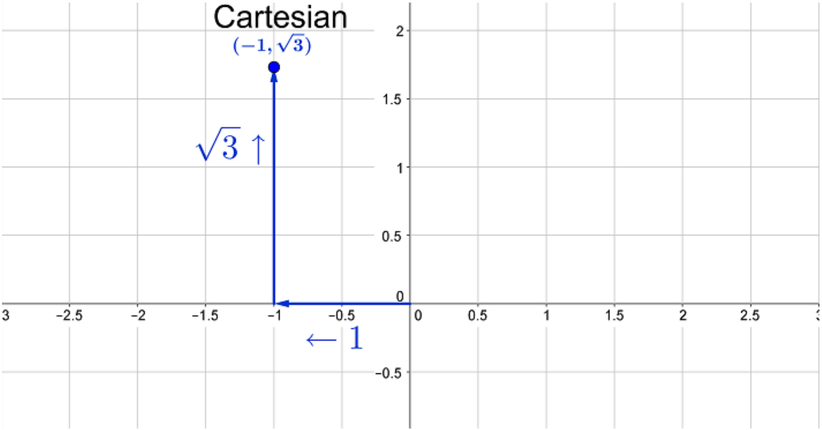
Polar Form

In Polar form we describe the position of a point using two measurements:

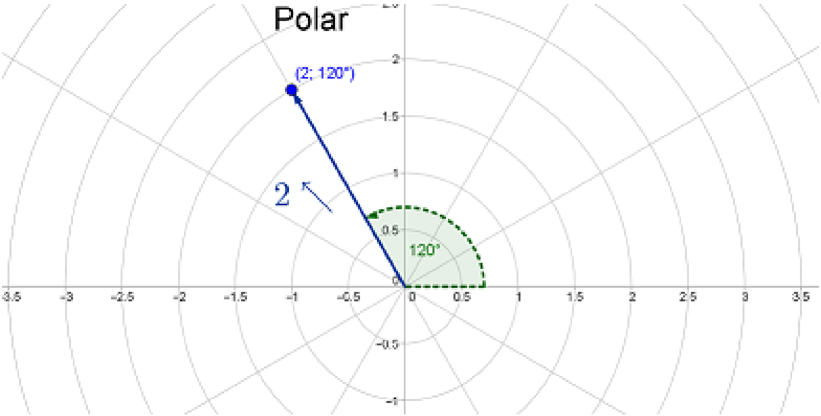
- The straight line distance of the point from the origin (the modulus).
- The angle the position of the point makes with the positive side of the x axis (the argument).



Cartesian Form

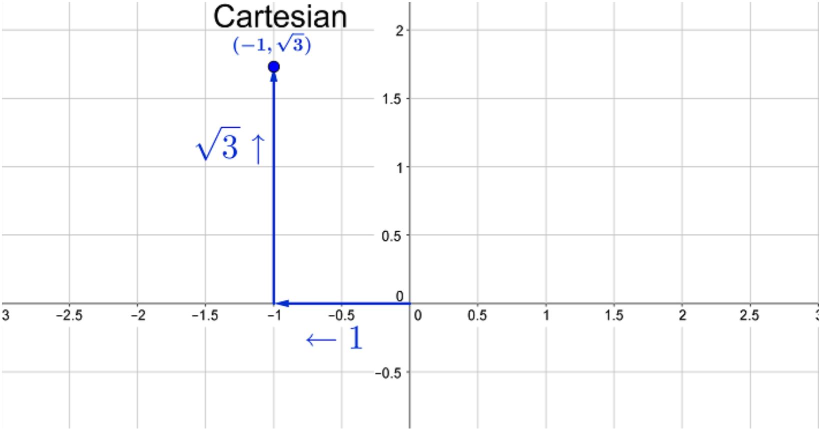


Polar Form



Polar Form

Write the complex no $z = -1 + \sqrt{3}i$ in polar form.



Polar Form of $z = -1 + \sqrt{3}i$

Modulus:

Argument:

Polar Form of $z = -1 + \sqrt{3}i$

Modulus:

$$z = \sqrt{a^2 + b^2}$$

Argument:

Polar Form of $z = -1 + \sqrt{3}i$

Modulus:

$$z = \sqrt{a^2 + b^2}$$

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

Argument:

Polar Form of $z = -1 + \sqrt{3}i$

Modulus:

$$z = \sqrt{a^2 + b^2}$$

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$r = \sqrt{4} = 2$$

Argument:

Polar Form of $z = -1 + \sqrt{3}i$

Modulus:

$$z = \sqrt{a^2 + b^2}$$

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$r = \sqrt{4} = 2$$

Argument:

$$\tan(A) = \frac{\sqrt{3}}{1}$$

Polar Form of $z = -1 + \sqrt{3}i$

Modulus:

$$z = \sqrt{a^2 + b^2}$$

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$r = \sqrt{4} = 2$$

Argument:

$$\tan(A) = \frac{\sqrt{3}}{1}$$

$$A = \tan^{-1}(\sqrt{3}) = 60^\circ$$

Polar Form of $z = -1 + \sqrt{3}i$

Modulus:

$$z = \sqrt{a^2 + b^2}$$

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$r = \sqrt{4} = 2$$

Argument:

$$\tan(A) = \frac{\sqrt{3}}{1}$$

$$A = \tan^{-1}(\sqrt{3}) = 60^\circ$$

$$\theta = 180 - 60 = 120^\circ \quad \left(\frac{2\pi}{3} radians\right)$$

Polar Form of $z = -1 + \sqrt{3}i$, $\quad r = 2$ and $\theta = 120$

We use the following format to present a complex number in polar form;

$$z = r(\cos\theta + i\sin\theta)$$

Polar Form of $z = -1 + \sqrt{3}i$, $\quad r = 2$ and $\theta = 120$

We use the following format to present a complex number in polar form;

$$z = r(\cos\theta + i\sin\theta)$$

If $z = -1 + \sqrt{3}i \quad r = 2$ and $\theta = 120$:

Polar Form of $z = -1 + \sqrt{3}i, \quad r = 2$ and $\theta = 120$

We use the following format to present a complex number in polar form;

$$z = r(\cos\theta + i\sin\theta)$$

If $z = -1 + \sqrt{3}i \quad r = 2$ and $\theta = 120$:

$$z = 2(\cos(120) + i\sin(120))$$

Polar Form of $z = -1 + \sqrt{3}i, \quad r = 2$ and $\theta = 120$

We use the following format to present a complex number in polar form;

$$z = r(\cos\theta + i\sin\theta)$$

If $z = -1 + \sqrt{3}i \quad r = 2$ and $\theta = 120$:

$$z = 2(\cos(120) + i\sin(120))$$

or in radians:

$$z = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$$

De Moivre's Theorem

De Moivre's Theorem tell us that if:

$$z = r(\cos\theta + i\sin\theta)$$

then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1 + \sqrt{3}i)^3$

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1+\sqrt{3}i)^3$

The first step is to write $-1+\sqrt{3}i$ in polar form.

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1+\sqrt{3}i)^3$

The first step is to write $-1+\sqrt{3}i$ in polar form.

$$-1+\sqrt{3}i = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$$

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1+\sqrt{3}i)^3$

The first step is to write $-1+\sqrt{3}i$ in polar form.

$$-1+\sqrt{3}i = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$$

Then using De Moivre's Theorem:

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1+\sqrt{3}i)^3$

The first step is to write $-1+\sqrt{3}i$ in polar form.

$$-1+\sqrt{3}i = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$$

Then using De Moivre's Theorem:

$$(1+\sqrt{3}i)^3 = 2^3(\cos\left(3\times\frac{2\pi}{3}\right) + i\sin\left(3\times\frac{2\pi}{3}\right))$$

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1+\sqrt{3}i)^3$

The first step is to write $-1+\sqrt{3}i$ in polar form.

$$-1+\sqrt{3}i = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$$

Then using De Moivre's Theorem:

$$\begin{aligned}(1+\sqrt{3}i)^3 &= 2^3(\cos\left(3\times\frac{2\pi}{3}\right) + i\sin\left(3\times\frac{2\pi}{3}\right)) \\ &= 8(\cos(2\pi) + i\sin(2\pi))\end{aligned}$$

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1+\sqrt{3}i)^3$

The first step is to write $-1+\sqrt{3}i$ in polar form.

$$-1+\sqrt{3}i = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$$

Then using De Moivre's Theorem:

$$\begin{aligned}(1+\sqrt{3}i)^3 &= 2^3(\cos\left(3\times\frac{2\pi}{3}\right) + i\sin\left(3\times\frac{2\pi}{3}\right)) \\ &= 8(\cos(2\pi) + i\sin(2\pi)) \\ &= 8(1+i(0))\end{aligned}$$

Example of Demoivre's Theorem

Use De Moivre's Theorem to find the value of $(-1+\sqrt{3}i)^3$

The first step is to write $-1+\sqrt{3}i$ in polar form.

$$-1+\sqrt{3}i = 2(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right))$$

Then using De Moivre's Theorem:

$$\begin{aligned}
(1+\sqrt{3}i)^3 &= 2^3(\cos\left(3\times\frac{2\pi}{3}\right) + i\sin\left(3\times\frac{2\pi}{3}\right)) \\
&= 8(\cos(2\pi) + i\sin(2\pi)) \\
&= 8(1+i(0)) \\
&= 8
\end{aligned}$$

Finding Roots with DeMoivre's Theorem

Solve the equation $z^3 = -27i$, using DeMoivre's Theorem.

Finding Roots with DeMoivre's Theorem

Solve the equation $z^3 = -27i$, using DeMoivre's Theorem.

$$z^3 = -27i$$

Finding Roots with DeMoivre's Theorem

Solve the equation $z^3 = -27i$, using DeMoivre's Theorem.

$$z^3 = -27i$$

$$z = (-27i)^{\frac{1}{3}}$$

Finding Roots with DeMoivre's Theorem

Solve the equation $z^3 = -27i$, using DeMoivre's Theorem.

$$z^3 = -27i$$

$$z = (-27i)^{\frac{1}{3}}$$

The first step is to write $-27i$ in polar form, then apply DeMoivre's Theorem:

Finding Roots with DeMoivre's Theorem

Solve the equation $z^3 = -27i$, using DeMoivre's Theorem.

$$z^3 = -27i$$
$$z = (-27i)^{\frac{1}{3}}$$

The first step is to write $-27i$ in polar form, then apply DeMoivre's Theorem:

$$-27i = 27(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right))$$

Finding Roots with DeMoivre's Theorem

Solve the equation $z^3 = -27i$, using DeMoivre's Theorem.

$$z^3 = -27i$$
$$z = (-27i)^{\frac{1}{3}}$$

The first step is to write $-27i$ in polar form, then apply DeMoivre's Theorem:

$$-27i = 27(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right))$$
$$(-27i)^{\frac{1}{3}} = 27^{\frac{1}{3}}(\cos\left(\frac{1}{3}\times\frac{3\pi}{2}\right) + i\sin\left(\frac{1}{3}\times\frac{3\pi}{2}\right))$$

Finding Roots with DeMoivre's Theorem

Solve the equation $z^3 = -27i$, using DeMoivre's Theorem.

$$z^3 = -27i$$
$$z = (-27i)^{\frac{1}{3}}$$

The first step is to write $-27i$ in polar form, then apply DeMoivre's Theorem:

$$-27i = 27(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right))$$
$$(-27i)^{\frac{1}{3}} = 27^{\frac{1}{3}}(\cos\left(\frac{1}{3}\times\frac{3\pi}{2}\right) + i\sin\left(\frac{1}{3}\times\frac{3\pi}{2}\right))$$
$$= 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

This is only our first solution! We have a total of three solutions to this question.

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

This is only our first solution! We have a total of three solutions to this question.

$$z^{\frac{1}{2}} \rightarrow 2 \text{ solutions} \rightarrow 180 \text{ degrees apart } (\pi \text{ radians})$$

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

This is only our first solution! We have a total of three solutions to this question.

$$z^{\frac{1}{2}} \rightarrow 2 \text{ solutions} \rightarrow 180 \text{ degrees apart } (\pi \text{ radians})$$
$$z^{\frac{1}{3}} \rightarrow 3 \text{ solutions} \rightarrow 120 \text{ degrees apart } (\tfrac{2\pi}{3} \text{ radians})$$

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

This is only our first solution! We have a total of three solutions to this question.

$$z^{\frac{1}{2}} \rightarrow 2 \text{ solutions} \rightarrow 180 \text{ degrees apart } (\pi \text{ radians})$$
$$z^{\frac{1}{3}} \rightarrow 3 \text{ solutions} \rightarrow 120 \text{ degrees apart } (\tfrac{2\pi}{3} \text{ radians})$$
$$z^{\frac{1}{4}} \rightarrow 4 \text{ solutions} \rightarrow 90 \text{ degrees apart } (\tfrac{\pi}{2} \text{ radians})$$

We can use this information to find our remaining solutions.

First solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

First solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

Since we have a total of 3 solutions, which are 120°, ($\frac{2\pi}{3}$ radians) apart, we need to add $\frac{2\pi}{3}$ radians to the angle from our first solution (the modulus is unchanged):

Second solution:

First solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

Since we have a total of 3 solutions, which are $120^\circ, (\frac{2\pi}{3}$ radians) apart, we need to add $\frac{2\pi}{3}$ radians to the angle from our first solution (the modulus is unchanged):

Second solution:

$$= 3(\cos\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{2\pi}{3}\right))$$

First solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

Since we have a total of 3 solutions, which are $120^\circ, (\frac{2\pi}{3}$ radians) apart, we need to add $\frac{2\pi}{3}$ radians to the angle from our first solution (the modulus is unchanged):

Second solution:

$$\begin{aligned} &= 3(\cos\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)) \\ &= 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)) \end{aligned}$$

Second Solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$

Second Solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$

To get our third solution, we will add $\frac{2\pi}{3}$ radians to the angle from our second solution (once again, the modulus is unchanged):

Third solution:

Second Solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$

To get our third solution, we will add $\frac{2\pi}{3}$ radians to the angle from our second solution (once again, the modulus is unchanged):

Third solution:

$$= 3(\cos\left(\frac{7\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{7\pi}{6} + \frac{2\pi}{3}\right))$$

Second Solution:

$$(-27i)^{\frac{1}{3}} = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$

To get our third solution, we will add $\frac{2\pi}{3}$ radians to the angle from our second solution (once again, the modulus is unchanged):

Third solution:

$$= 3(\cos\left(\frac{7\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{7\pi}{6} + \frac{2\pi}{3}\right))$$
$$= 3(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right))$$

So we now have our three solutions:
$z = (-27i)^{\frac{1}{3}} =:$

So we now have our three solutions:
$z = (-27i)^{\frac{1}{3}} =:$

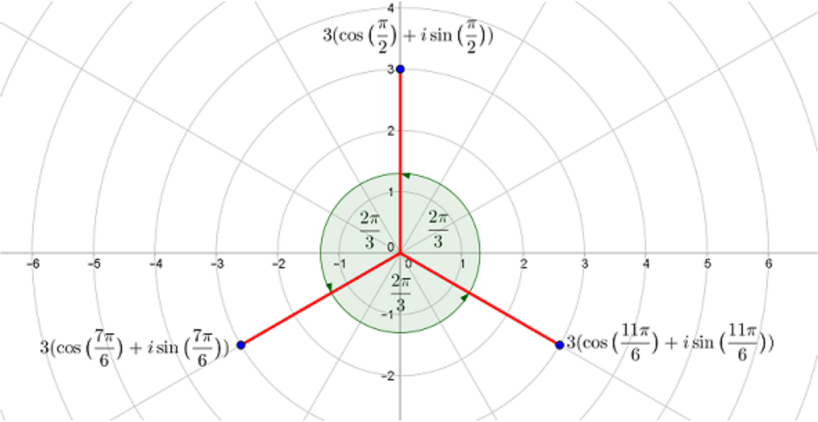
$$z_1 = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

So we now have our three solutions:
$z = (-27i)^{\frac{1}{3}} =:$

$$z_1 = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$
$$z_2 = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$

So we now have our three solutions:
$z = (-27i)^{\frac{1}{3}} =:$

$$z_1 = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$
$$z_2 = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$
$$z_3 = 3(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right))$$


$3(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$
$3(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6}))$
$3(\cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6}))$
$\frac{2\pi}{3}$
$\frac{2\pi}{3}$
$\frac{2\pi}{3}$
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
4 3 2 1 -1 -2

Converting these back into Cartesian form we get:

$$z_1 = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$$

$$z_2 = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$

$$z_3 = 3(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right))$$

Converting these back into Cartesian form we get:

$$z_1 = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)) \qquad = 3i$$

$$z_2 = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right))$$

$$z_3 = 3(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right))$$

Converting these back into Cartesian form we get:

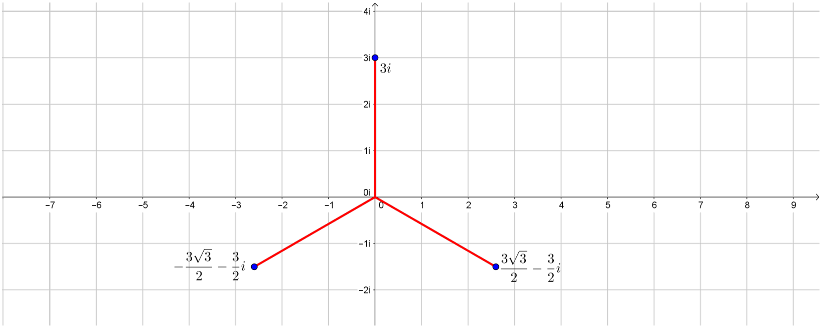
$$z_1 = 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)) = 3i$$

$$z_2 = 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

$$z_3 = 3(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right))$$

Converting these back into Cartesian form we get:

$$
\begin{aligned}
z_1 &= 3(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)) &&= 3i \\
z_2 &= 3(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)) &&= -\frac{3\sqrt{3}}{2} - \frac{3}{2}i \\
z_3 &= 3(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)) &&= \frac{3\sqrt{3}}{2} - \frac{3}{2}i
\end{aligned}
$$



4i
3i
2i
1i
0i
-1i
-2i
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
$3i$
$-\frac{3\sqrt{3}}{2}-\frac{3}{2}i$
$\frac{3\sqrt{3}}{2}-\frac{3}{2}i$