Complex Numbers

1 Basic Arithmetic

1. Let $z_1 = 7 + 5i$ and $z_2 = 2 + i$. Evaluate each of the following:
   i. $z_1 + z_2$
   ii. $z_1 - z_2$
   iii. $3z_1 + 2z_2$
   iv. $2z_1 - 3z_2$
2. Let $z_1 = 3 - 2i$ and $z_2 = 2 + 4i$. Evaluate each of the following:
   i. $z_1 + z_2$
   ii. $z_1 - z_2$
   iii. $z_1 + 3z_2$
   iv. $2z_1 - 5z_2$
3. Let $z_1 = -1 + 2i$ and $z_2 = 2 + 3i$. Evaluate the following:
   i. $2z_1 + z_2$
   ii. $2z_1 - z_2$
   iii. $2z_2 - 3z_1$
   iv. $z_2 - z_1$

2 Modulus and Argand Diagram

1. Evaulate each of the following:
   i. $|8 - 6i|$
   ii. $|5 - 12i|$
   iii. $|7 - 24i|$
   iv. $|8 + 15i|$
   v. $|9 - 40i|$
   vi. $|-3 + 4i|$
2. Evaluate each of the following:
   i. $|11 + 60i|$

ii. $|3 - 4i|$

iii. $|12 + 35i|$

iv. $|13 - 84i|$

v. $|-33 + 56i|$

vi. $|6 - 8i|$

3. If $|11 + 2i| = |10 + ki|$, then find two possible values of k, where $k \in R$.

4. If $|8 + ki|$=10, then find two possible values of k, where $k \in R$.

5. The complex numbers u, v and w are related by the equation $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$.
Given that $v = 3 + 4i$ and $w = 4 - 3i$, find u in the form $x + yi$.

6. If $z = 2 - 2i$, find $|z|, |2z|$ and $|3z|$. Is $2|z| = |2z|$?
Explain your answer.

7. Let $z_1 = s + 8i$ and $z_2 = t + 8i$, where $s, t \in R$ and $i^2 = -1$.
   i. Given that $|z_1| = 10$, find the value of s.
   ii. Given that $|z_2| = 2|z_1|$, find the value of t.

3 Multiplication

1. Evaluate the following products:
   i. $7i(3 + 5i)$
   ii. $i(3 - i)$
   iii. $2(7 + i)$
   iv. $3(1 + 5i)$
   v. $7(2 - i)$
   vi. $-2(4 + i)$

2. Write the following products in the form $a + bi$:
   i. $(2 + 7i)(3 - 5i)$
   ii. $(1 + 4i)(2 + 5i)$
   iii. $(6 + i)(-2 + 3i)$
   iv. $(2 + 3i)(2 - 3i)$
   v. $(3 + 4i)(3 - 4i)$
   vi. $(1 + i)(7 - 3i)$

3. Write the following products in the form $a + bi$:
   i. $3i(2 + 4i)$
   ii. $(1 - i)(1 + i)$
   iii. $5(6 - i)$

iv. $(-2-2i)(-2+2i)$
v. $(7+5i)(2+i)$
vi. $(3-2i)(7i)$

4. $z_1 = 3+i$
   i. Find z_2 if $z_2 = iz_1$
   ii. Plot z_1 and z_2 on an Argand diagram.
   iii. Describe the transformation that maps z_1 onto z_2.

5. $z_1 = -3+2i$
   i. Find z_2 if $z_2 = iz_1$.
   ii. Plot z_1 and z_2 on an Argand diagram.
   iii. Describe the transformation that maps z_1 onto z_2.

6. $z_1 = -3 = 4i$
   i. Find z_2 if $z_2 = -iz_1$
   ii. Plot z_1 and z_2 on an Argand diagram.
   iii. Describe the transformation that maps z_1 onto z_2.

7. $z_1 = -3+4i$
   i. Write, in the form $a+bi$, the product z_1z_2
   ii. Evaluate $|z_1|, |z_2|$ and $|z_1z_2|$.
   iii. Show that $|z_1||z_2| = |z_1z_2|$.

4 Conjugate

1. $z_1 = 7+5i$ and $z_2 = 2+i$. Find :
   i. $\bar{z}_1$
   ii. $\bar{z}_2$
   iii. $z_1 + \bar{z}_1$
   iv. $z_2 + \bar{z}_2$

2. $z_1 = 5+2i$ and $z_2 = 3-4i$. Find:
   i. $\bar{z}_1$
   ii. $\bar{z}_2$
   iii. $\bar{z}_1 + \bar{z}_2$
   iv. $z_1 + z_2$
   v. $\overline{z_1 + z_2}$

3. $z_1 = -1+2i$ and $z_2 = 2+3i$. Find :
   i. $\bar{z}_1$

ii. $\bar{z}_2$

iii. $\bar{z}_1 + \bar{z}_2$

iv. $z_1 + z_2$

v. $\overline{z_1 + z_2}$

5 Division

1. Write the following in the form $p + qi, p, q \in Q$:
   i. $\frac{6+3i}{2}$
   ii. $\frac{15-20i}{5}$
   iii. $\frac{24+12i}{4}$
   iv. $\frac{16-8i}{8}$
   v. $\frac{5+12i}{7}$

2. Write the following in the form $p + qi, p, q \in R$:
   i. $\frac{5+5i}{1+2i}$
   ii. $\frac{1-5i}{1-i}$
   iii. $\frac{10}{1-3i}$
   iv. $\frac{5}{1+2i}$
   v. $\frac{1+3i}{1+i}$

6 Quadratic Equations with complex roots

1. Solve the equation $z^2 + 4z + 13 = 0$, giving your answers in the form $a + bi, a, b \in R$

2. Evaluate the following:
   i. $(1 - 2i)^2$
   ii. $-2(1 - 2i)$

   Hence show that $1 - 2i$ is the root of the equation $z^2 - 2z + 5$

3. Show that the roots of the equation $z^2 - 2z + 10 = 0$ are complex.

4. Solve the equation $z^2 - 8z + 17 = 0$, giving your answers in the form $a + bi, a, b \in R$

5. If $3 - 2i$ is a root of the equation $z^2 + kz + 13 = 0$, where $k \in R$, find the value of k and write down the second root.

6. One root of the equation $z^2 + (-1 + 5i)z + p(2 - i) = 0$ is $1 + 2i$.
   Find
   i. the value of p
   ii. the other root of the equation.

7. Use the $quadratic formula$ to solve the equation $z^2 - (2+2i)z + 2i - 1 = 0$, giving your answers in the form $a + ib$

8. Solve the equation $z^2 - z(1+4i) + 2i - 4 = 0$, expressing your answers in the form $a + ib$

7 Polar Form

1. Express each of the following complex numbers in polar form
   i. $1 + i$
   ii. $\sqrt{3} + i$
   iii. $-2 + i\sqrt{2}$
   iv. $-2 - i\sqrt{2}$
   v. $4i$
   vi. -5
   vii. $-3i$
   viii. $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

2. Simplify each of the following, giving your answer in the form $r(\cos\theta + i\sin\theta)$.
   i. $(1 + i\sqrt{3})^2$
   ii. $\frac{-2}{-\sqrt{3}+i}$

3. Express each of the following in polar form:
   i. $2i$
   ii. $-3 - i\sqrt{3}$
   iii. $\frac{2}{-1+i}$

4. Solve the equation $z^2 - 2z + 2 = 0$ and express your answer in the form $r(\cos\theta + i\sin\theta)$.

8 de Moivre's Theorem

1. Change each of the following to polar form and then use de Moivre's theorem to express your answers in the form $a + bi$:
   i. $(1 - i)^4$
   ii. $(1 + i\sqrt{3})^3$
   iii. $(-2 - 2i)^4$

2. Simplify
   i. $(3 - \sqrt{3}i)^6$
   ii. $(2 + 2i\sqrt{3})^6$

3. Express $\frac{\sqrt{3}+i}{1+i\sqrt{3}}$ in the form $r(\cos\theta + i\sin\theta)$.
   Hence, evaluate $(\frac{\sqrt{3}+i}{1+i\sqrt{3}})^6$.

4. Use de Moivre's theroerm to write the following in the form $a + bi$
   i. $(2 + 2i)^8$
   ii. $(1 - i)^{16}$
   iii. $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^{13}$
   iv. $(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{11}$
   v. $(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{60}$
   vi. $(-\frac{\sqrt{3}}{2} + \frac{1}{2}i)^6$
   vii. $(-2 - 2i)^9$
   viii. $(-1 + i)^{13}$

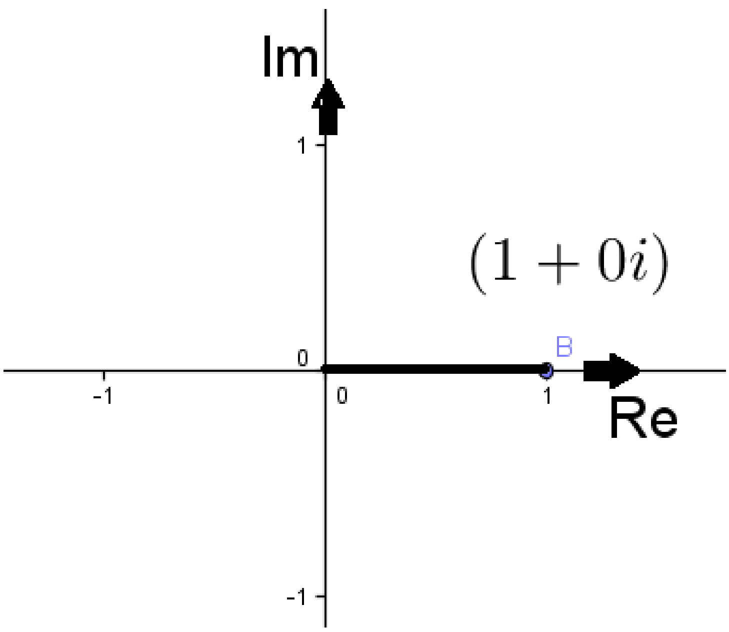
9 Finding the n^{th} root

1. Use de Moivre's theorem to solve the equation $z^3 = 8$

2. Find the values of z for which $z^3 = -8$, giving your answer in $a + bi$ form.

3. Plot the point $2 + 2\sqrt{3}i$ on an Argand Diagram.
   Use this diagram to express $2 + 2\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$.
   Hence find the solution set of $z^2 = 2 + 2\sqrt{3}i$

4. The complex number $z = 1$ is plotted on this Argand diagram.
   Write down the modulus and argument of this number.



   i. Express $z = 1$ in general polar form and hence find the cube roots of unity, that is, find the values of z for which $z = 1^{\frac{1}{3}}$.

ii. Prove that the sum of these roots is zero.

5. Find the cube roots of $27i$

6. Use de Moivre's Theorem to solve,

   i. $z^2 = 1 + \sqrt{3}i$

   ii. $z^2 = 2 - 2\sqrt{3}i$

   iii. $z^2 = 4i$

7. Use de Moivre's theorem tp find, in polar form, the five roots of the equation $z^5 = 1$. Choose one of the roots w, where $w \neq 1$, and prove that $w^2 + w^3$ is real.