Probability Distributions
Revision Series 2016

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Binomial Experiment

If I roll a die 5 times, what is the probability that I roll exactly 3 twos.
The probability of getting a two is $\frac{1}{6}$.
The probability of not getting a two is therefore $1 - \frac{1}{6} = \frac{5}{6}$.
One way I could roll exactly 3 twos is on the first three goes, then roll a different number on the final 2 goes. The probability of this happening is:

$$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$$
$$\left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2 = \frac{25}{7776}$$

However this is not the only way to roll exactly three 2's.

Binomial Experiment

If I roll a dice 5 times, how many ways can I roll 3 twos? This takes us back to our counting techniques. The number of ways to roll 3 twos is:

$$\binom{5}{3} = 10$$

Therefore the probability that I roll exactly 3 twos is:

$$\binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{250}{7776}$$

Binomial Distribution

The above is an example of what is known as a **binomial experiment**

- Fixed number of trials
- Each trial has only two possible outcomes
- The outcomes of the trials are independent from each other
- The probability of *success* remains the same for each trial

Binomial Distribution

In a binomial experiment, the probability of exactly r successes in n trials is:

$$P(r) = \binom{n}{r}.p^r.q^{n-r}$$

where p is the probability of success in an individual trial, and q is $1 - p$.

Example

An exam has 10 multiple choice questions, with 4 choices in each question. If a student randomly guesses each of the question, what is the probability the student gets exactly 7 correct?

$$n = 10$$
$$p = .25$$
$$q = .75$$
$$r = 7$$

$$P(7) = \binom{10}{7}.(.25)^7.(.75)^3$$

$P = .003$

Example

A coin is flipped until 8 heads appear. Find the probability that this takes exactly 16 flips.

The only way this can happen is if there are exactly 7 heads during the first 15 flips, **and** the 16^{th} flip gives a head:

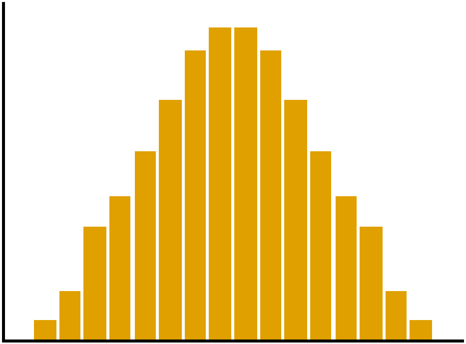
$$P = \binom{15}{7}.(.5)^7.(.5)^8 \times .5$$

$$P = .098$$

The probability of getting the 8^{th} on the 16^{th} flip is .098

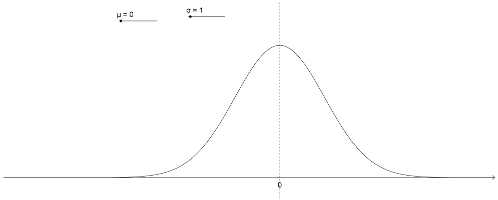
Normal Distribution

Variables that are **normally distributed** assume a shape such as the one below.



Bell Curve

For continuous variables that are approximately normally distributed we can graph the probabilities of their values with a curve, known sometimes as the bell curve.



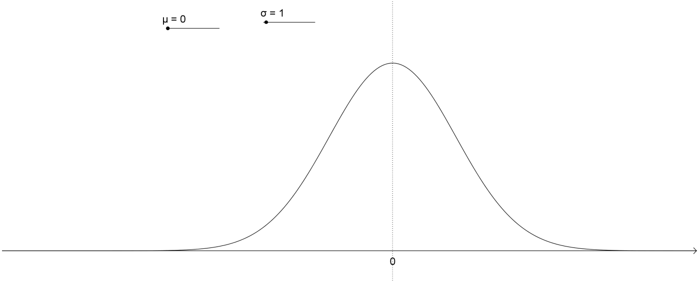
The heights of adult men and cholesterol level of adults are variables which are approximately normally distributed.

Properties

- The normal distribution is bell shaped.
- The mean, median and mode are equal, and located at the centre of the distribution.
- The normal distribution curve is symmetric about the mean.
- The total area underneath the curve is 1.

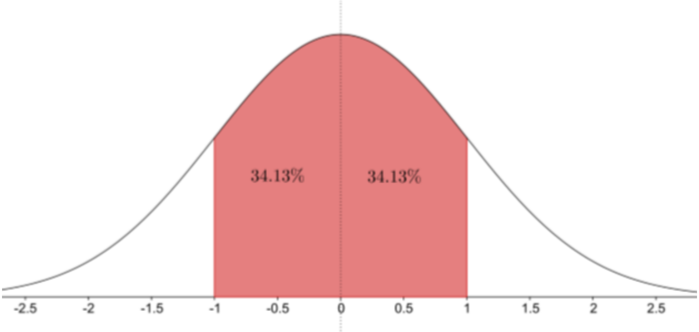
Standard Normal Distribution

The **standard normal distribution** has a mean of 0, and a standard deviation of 1. We use this distribution as the basis for calculations of probabilities of variables that are approximately normally distributed.

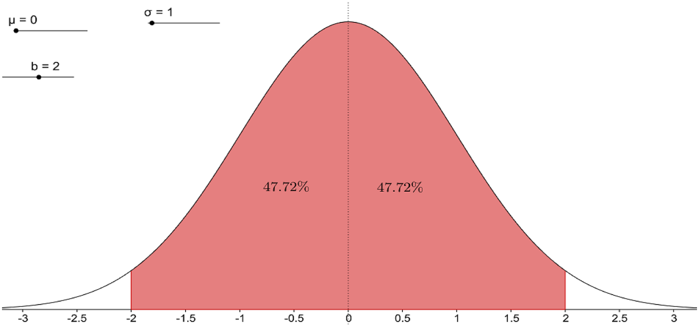


Empirical Rule

One standard deviation each side of the mean contains about 68.26% of the data. The area from one s.d. below the mean to one s.d. above the mean therefore is .6826.



Two standard deviations each side of the mean contain about 95.44% of the data.



Similarly three standard deviations each side of the mean contain about 99.7% of the data

Z-tables

A **z value** is the value on the horizontal axis of the standard normal distribution. We use the **z tables** (log table pg. 36-37) to find the corresponding areas for these z values. In your log tables, the z values are the values along the side, and the values inside in the table are the corresponding areas.

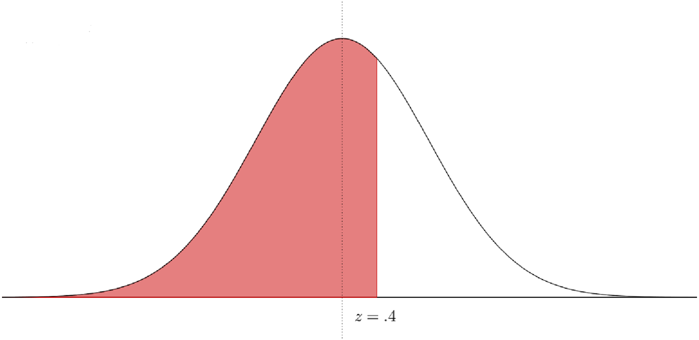
The areas underneath the curve correspond to the associated probability.

It is important to notice that **your** z tables only give areas for z values greater than 0.

It is also important to notice that your tables give you the area **up to** the z-value.

Example

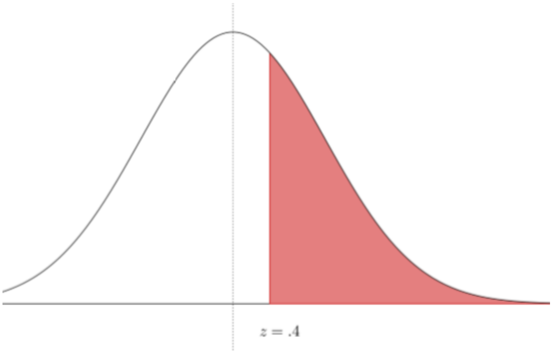
Find $P(z \leq .4)$



When we look up .4 in the z-tables, we get an answer of .6554.

Example

Find $P(z > .4)$



We know that the whole area underneath the curve is 1, so
$P(z > .4) = 1 - .6554 = .3446$

Real Variables

Since we cannot have a separate set of tables for every variable with a different mean and standard deviation, we need some way to convert requested values into z-values to look up in the standard normal tables. If a variable is normally distributed, and we are interested in calculating probabilities of certain values, the formula we need to find the corresponding z values is:

$$z = \frac{x-\mu}{\sigma}$$

where x is the value we are interested in, μ is the mean, and σ is the standard deviation.

Example

I.Q. is approximately normally distributed with a mean of 100 and a standard deviation of 15. If a person is selected at random, what is the probability they have an I.Q. greater than 110?

$$x = 110, \mu = 100, \sigma = 15$$

$$z = \frac{110-100}{15} = .67$$

So $P(x > 110) \rightarrow P(z > .67)$

Looking up .67 in our z tables we get .7486. Remember, this is $p(z \leq .7486)$ so $p(z > .67) = 1 - .7486 = .2514$

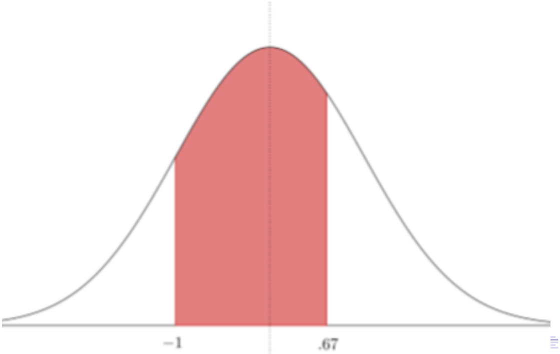
So the probability that someone will have an I.Q. greater than 110 is .2514.

What is the probability that a person will have an I.Q. between 85 and 110?

$z = \frac{85-100}{15} = -1$

$z = \frac{110-100}{15} = .67$

So what we are looking for is: $P(-1 \leq z \leq .67)$



$P(-1 \leq z \leq .67)$

There are a few ways to calculate this.
I will break it up into $p(-1 \leq z \leq 0)$ and $p(0 \leq z \leq .67)$ and add the two areas together.

$p(-1 \leq z \leq 0)$:

$$p(z \geq -1) = .8413$$

$$.8413 - .5 = .3413 \qquad p(-1 \leq z \leq 0) = .3413$$

$p(0 \leq z \leq .67)$

$$p(z \leq .67) = .7486$$

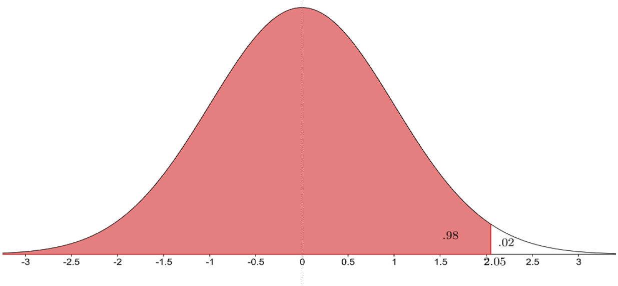
$$.7486 - .5 = .2486$$

$p(0 \leq z \leq .67) = .2486$

so, $p(-1 \leq z \leq .67) = .2486 + .3413 = .5899$ The probability of a person having an I.Q. between 85 and 110 is .5899.

Working Backwards

To be eligible for Mensa one must be in the top 2% in terms of I.Q. What is the minimum I.Q. needed to be eligible for Mensa?



What z value gives us an area of .98?

We search the tables for the area closest to .98 and note its corresponding z value.

The z-value of 2.05 gives and area of .9798, this is the closest approximation. We now must find what value (in terms of I.Q.) corresponds to a z-value of 2.05.

$$2.05 = \frac{x-100}{15}$$

$$30.75 = x - 100$$

$$x = 130.75$$

One would need a minimum I.Q. of 131 to be eligible for Mensa.