Probability
Revision Series 2016

LEAMY MATHS COMMUNITY

Stephen King

April 1 , 2016

Fundamental Principal of Counting

If one event has m possible outcomes, and another event has n possible outcomes, the total number of possible outcomes including both events is $m \times n$.

Example I want to buy a car. There is 3 choices of car, and four colours to choose from for each. Therefore I have:

$$3 \times 4 = 12$$

12 possible cars to choose from.
This can also be extended to more than just two events.

Example

I have three hats, five t-shirts, four pairs of jeans, and two pairs of shoes.

From this combination I can create:

$$3 \times 5 \times 4 \times 2 = 120$$

120 different possible outfits.

Arranging

We also need to know how many ways there are to arrange n objects, eg., how many ways can the letters TOP be arranged?

One way to solve this is to create a **sample space** of all possible answers:

TOP TPO POT PTO OPT OTP

We can see from this that there are 6 different ways to arrange the letters TOP.

Factorials

Creating sample spaces can be time consuming and inefficient. The mathematical solution to arranging n objects is $n!$

$n!$ means $n(n-1)(n-2)....(2)(1)$

Eg. $5!$ is $5 \times 4 \times 3 \times 2 \times 1 = 120$

Make sure you can find the $n!$ button on your calculator.

Example

An indoor soccer team which has 6 players lines up for a photo. How many ways can they be arranged.

$$6! = 720$$

Arranging r objects, from a selection of n objects

A news show usually broadcasts three news reports per show, a lead story, a second story and a closing story. The director has seven stories to choose from. How many possible ways can the new broadcast be set up.

We have to arrange 3 objects from a selection of 7. We can write this as ${}^7P_3$
${}^7P_3 = 210$

Permutations

The previous question was an example of a **permutation** of n objects, taking r objects at a time. The formula is:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Example

In a board of directors composed of eight people, how many ways can a chairperson, secretary and treasurer be selected?

$$^8P_3 = 336$$

Example

Another method to answer this is to use the multiplication rule:

First Choice	$\times$	Second Choice	$\times$	Third Choice
8	$\times$	7	$\times$	6

$$8 \times 7 \times 6 = 336$$

Choosing/Combinations

Combinations are used when the order of arrangement of the selection is not important.

Example - How many ways can a committee of 4 people be chosen from a group of 9?

$$^9C_4 = 126$$

Combinations

The number of combinations of r objects, selected from n objects is given by ${}^{n}C_{r}$.

$$ {}^{n}C_{r} = \frac{n!}{(n-r)!r!} $$

Example

How many ways is there to select 3 hats from a box of 8 hats, disregarding order of selection.

$$^8C_3 = 56$$

Example

It is important to be able to use the multiplication rule with combinations.

In a train yard there are 4 tank cars, 12 boxcars and 7 flatcars. How many ways can a train be made up consisting of 2 tank cars, 5 boxcars and 3 flat cars (order not important)?

$$^4C_2 \times ^{12}C_5 \times ^7C_3 = 166320$$

Introduction

Probability is the study of the likeliness of certain events occurring following a trial.

A **trial** is the act of doing an experiment that has different possible outcomes, eg. Flipping a coin, picking a card, rolling a dice etc.

Outcomes are the possible results of a trial, and an **event** is the occurrence of one or more outcomes.

The probability of an event must lie between 0 and 1. For example the probability of getting a head after flipping a coin is 0.5.

Properties in Probability

- $0 \leq P(E) \leq 1$, where $P(E)$ is the probability of any event E.
- If p is the probability of an event E happening, the probability of E not happening is $1 - p$.
- If p is the probability of an event E happening, and q is the probability of an **independent** event F happening, the probability that E and F will happen in successive trials is $p \times q$. This is known as the **multiplication rule**.

In simple cases like dice or coins it is quite straight forward to work out certain probabilities. However, many events are not so simple. For example, what is the probability of a certain basketball player scoring a free throw?

To calculate this we will use experimental probability techniques, otherwise known as relative frequency. If the player has scored 78 out of 90 free throws this season, his relative frequency for scoring free throws is:

$$\frac{78}{90} = .8667$$

Relative frequency is often the best estimate of the probability of an event, as long as there has been large number of trials to draw data from.

The best estimate for the probability that the player makes the free throw is therefore .8667

The probability of him missing the free throw is $1 - .8667 = .1333$

Relative Frequency

Relative frequency is:

$$\frac{\textit{No. of times event occurred in a trial}}{\textit{Total amount of trials}}$$

The probability of an event E happening is denoted by $P(E)$ and can be calculated by:

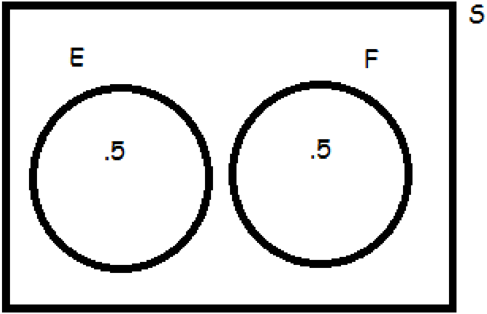
$$P(E) = \frac{number\ \ of\ \ ways\ \ the\ \ event\ \ can\ \ occur}{number\ \ of\ \ all\ \ possible\ \ outcomes\ \ in\ \ the\ \ trial} = \frac{\#E}{\#S}$$

S is the sample space,i.e. the set of all possible outcomes.

Venn Diagrams

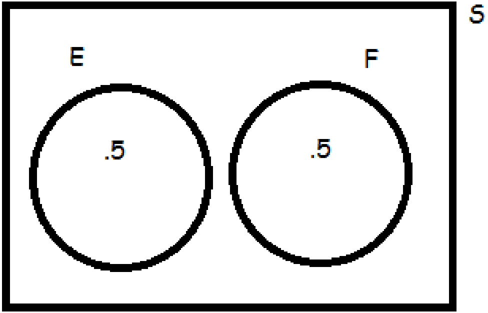
Using Venn Diagrams can help illustrate the probabilities of various events in a sample space. The following diagram below displays the probabilities of getting a heads or tails when flipping a fair coin.

E is the event of getting a heads, F is the event of getting a tails.



S, the sample space is similar to the universal set.
$P(E) = .5$ and $P(F) = .5$
It is important that all of the probabilities in the sample space add to 1.

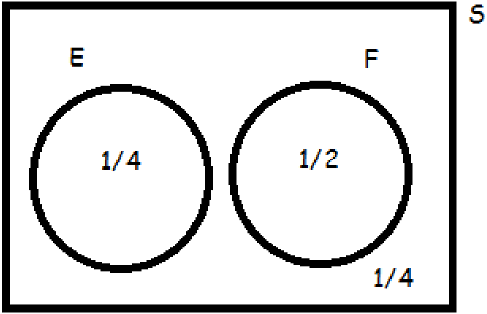
Mutually Exclusive Events



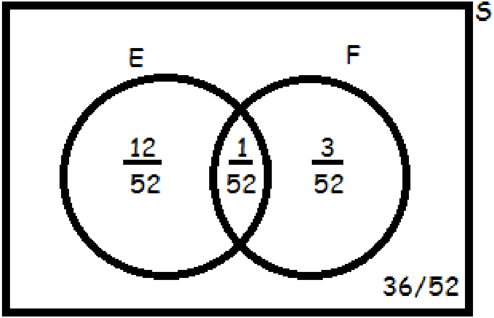
E and F are **mutually exclusive** events, since $E \cap F = \emptyset$

Mutually Exclusive Events

For example, if we pick a card from a pack. E is the event of getting a clubs and F is the event of getting a red card. E and F are mutually exclusive.



If E is the event of getting a clubs and F is the event of getting an Ace, these events are **not** mutually exclusive.



Here $P(E \cap F) = \frac{1}{52}$

Addition Rule (OR)

$$P(E \ \ or \ \ F) = P(E \cup F) = P(E) + P(E) - P(E \cap F)$$

Example

If I draw a card from a deck, what is the probability that I get a Clubs **or** an Ace.

$$P(E \ \ or \ \ F) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52}$$

$$P(E \ \ or \ \ F) = \frac{16}{52} = \frac{4}{13}$$

Independent Events

Two events are said to be independent of each other if the outcome of one event does not effect the outcome of the other event. An example of this would be flipping a coin and rolling a dice. The result of the coin flip in no way effects the result of the dice roll.

Multiplication Rule -If two events are independent, the probability of both occurring is:

$$P(A \ \ and \ \ B) = P(A) \times P(B)$$

Example

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die:

$$\mathrm{P}(\text{head } \textbf{and } 4) = P(head) \times P(4)$$

$$\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Example

A card is drawn from a deck and then **replaced**; then a second card is drawn. Find the probability of getting a Queen (Q) and then an Ace (A):

$$\text{P(Q and A)} = P(Q) \times P(A)$$

$$\frac{4}{52} \times \frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$$

Independent Events

Two events A and B are independent if:

$$P(A) \times P(B) = P(A \cap B)$$

Counting

Sometimes we have to use counting methods to figure out probabilities. Remember the probability of an event E happening is denoted by $P(E)$ and can be calculated by:

$$P(E) = \frac{number \quad of \quad ways \quad the \quad event \quad can \quad occur}{number \quad of \quad all \quad possible \quad outcomes \quad in \quad the \quad trial} = \frac{\#E}{\#S}$$

S is the sample space,i.e. the set of all possible outcomes. Sometimes we can use counting methods to calculate the number of ways an event can occur, or the number of possible outcomes.

Example

Three cards are chosen at random from a standard pack. Find the probability that:

1. All three are Kings.
2. None of the three cards are Kings.
3. At least one is a King.
4. Exactly two are spades.

All three are Kings

$$\frac{\textit{no. of ways of choosing 3 Kings out of 4}}{\textit{no. of ways of choosing 3 cards from 52}}$$

no. of ways of choosing 3 Kings out of 4:

$$\binom{4}{3} = 4$$

no. of ways of choosing 3 cards from 52:

$$\binom{52}{3} = 22100$$

so therefore the probability of all three being Kings is:

$$\frac{4}{22100} = \frac{1}{5525}$$

Probability that none are Kings

no. of ways of choosing 3 non-Kings:

$$\binom{48}{3} = 17296$$

no. of ways of choosing three cards from 52:

$$\binom{52}{3} = 22100$$

P(none is a King):

$$\frac{17296}{22100} = \frac{4324}{5525}$$

At Least One!

To calculate the probability of getting **at least one** we use the following method:

$$1 \text{ - } P(\text{none}) = P(\text{at least one})$$

At Least One is a King

1 - P(none is a King) $=$ P(at least one is a King):

$$1 - \frac{4324}{5525} = \frac{1201}{5525}$$

Exactly Two Are Spades

no. of ways of choosing 2 Spades **and** 1 non-Spade.

$$\binom{13}{2} \times \binom{39}{1} = 3042$$

no. of ways of choosing 3 cards from 52:

$$\binom{52}{3} = 22100$$

P(exactly two are Spades):

$$\frac{3042}{22100} = \frac{117}{850}$$

Conditional Probability

For two events, A and B, the probability that the second event B occurs **given** that the first event A has occurred can be calculated by:

$$P(B|A) = \frac{P(AandB)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

Example

The probability that Sam parks in a no-parking zone **and** gets a parking ticket is .06 $(N \cap T)$, and the probability that Sam parks in a no parking zone is .2 (N): If Sam parks in a no parking zone, find the probability he will get a ticket.

$$P(T|N) = \frac{P(N \cap T)}{P(N)} = \frac{.06}{.2} = .3$$

So Sam has a .3 probability of getting a parking ticket, given that he parked in a no-parking zone

Example

A bag contains 11 balls numbered 1 to 11. A ball is drawn from the bag and the number is noted.
Let A = the number is even
Let B = the number is greater than 6.

Find $P(A|B)$:

$$P(A) = \frac{5}{11} \quad P(B) = \frac{5}{11}$$

$$P(A \cap B) = \frac{2}{11}$$

$P(A|B)$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{11}}{\frac{5}{11}} = \frac{2}{5}$$

Example

Expected value can be defined by:

$$E(X) = \Sigma((x).P(x))$$

We multiply the value by their probability then total up the results. The expected value can be seen as 'the average' value received over many trials.

Example

You bet €5 to enter a game of chance.
The probability of winning €2 is $\frac{1}{2}$.
The probability of winning €4 is $\frac{1}{4}$.
The probability of winning €6 is $\frac{1}{4}$.
What is the expected return?

$$E(X) = \Sigma((x).P(x))$$
$$E(X) = 2(\tfrac{1}{2}) + 4(\tfrac{1}{4}) + 6(\tfrac{1}{4}) = 3.5$$

The expected return is €3.50
It costs five euros to enter the game...this is an example of why in gambling they say 'the house always wins'.