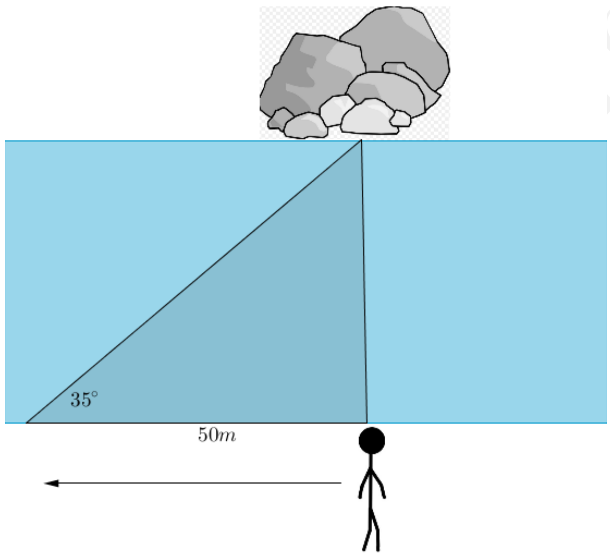
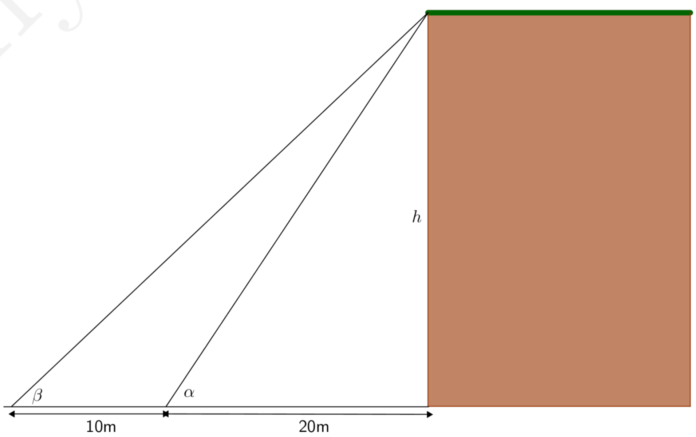
6 Trigonometry

6.1 Sine, Cosine and Tangent

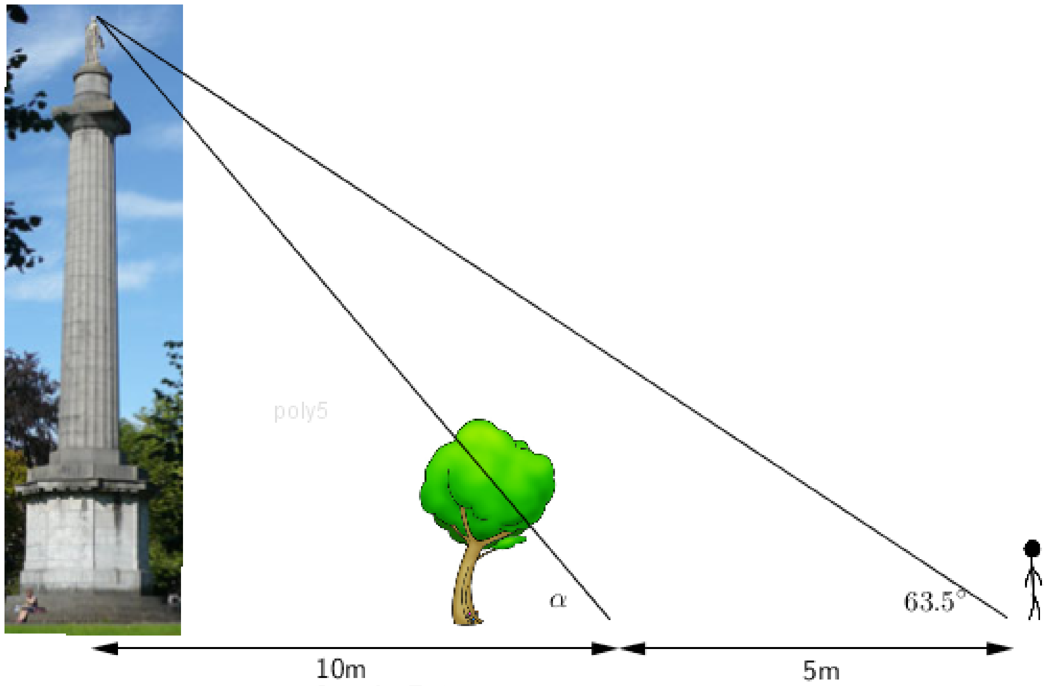
1. The stickman then comes across a river and wants to approximate its width. He begins by standing directly opposite a large group of boulders, then walks 50m downstream and using a clinometer measures the angle 35°.
   (i) What is the width of the river (to the nearest m)?
   (ii) How far is he now standing from the boulders?



2. Calculate the angles α and β if the height of the cliff is 50m.

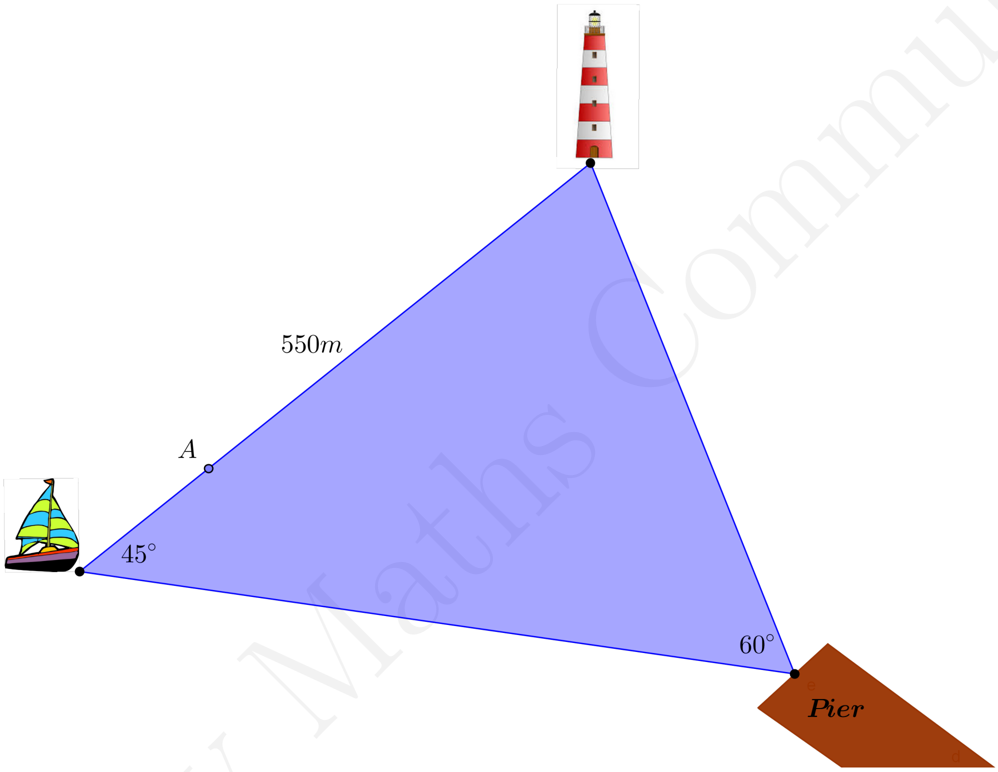


3. A student from Leamy Maths Community want's to approximate the height of the Rice's Memorial Column in the nearby People's Park. The student begins by standing directly under the monument, and measuring ten metres out. Unfortunately, from this position there is a tree blocking the view of the top, so the student walks another 5m and measures the angle of inclination 63.5°.
(i) How tall is the column?
(ii) If the tree was chopped down, what would the original attempted angle of inclination have measured (i.e. α).

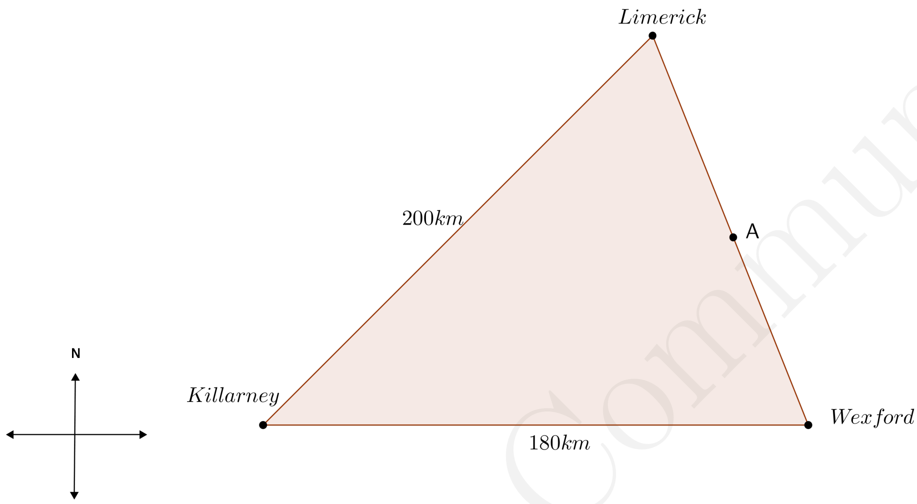


6.2 Sine/Cosine Rule in Context

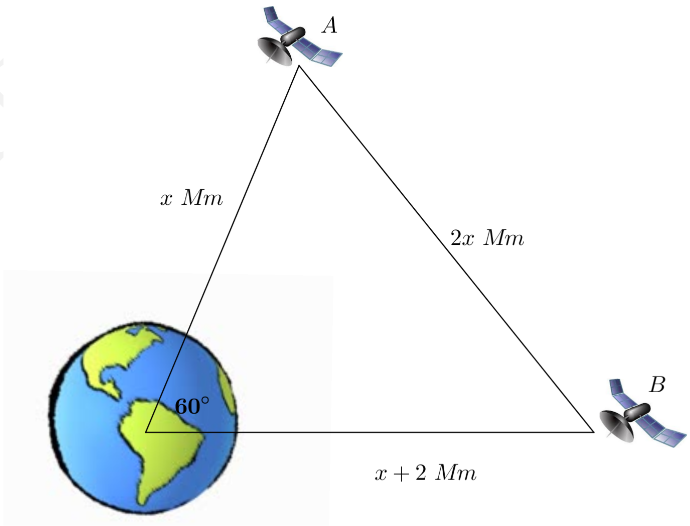
1. Standing at the edge of a pier, an observer watches a boat head directly towards a lighthouse. Given that the boat is 550m away from the lighthouse, using the angle measurements seen in the figure below, calculate(to the nearest metre or second),
   (i) How far the boat is from the pier?
   (ii) How far the lighthouse is from the pier?
   (iii) The boat is travelling at 14km/hr, how long until it reaches the lighthouse?
   (iv) 50m into the boats journey (point A), the weather worsens and the captain decides to head for the pier. What distance does the boat now have to travel?
   (v) With the wind the boat can now travel at 20km/hr, how long until it reaches dry land?



2. A group of swallows from Killarney cannot decide whether to fly to Wexford or Limerick for the weekend. Given that Wexford is twenty kilometers closer, they head due East. Upon realising there isn't much to do in Wexford, they decide to make the journey towards Limerick.
(i) Given that Limerick is exactly North-East of Killarney, how far is Wexford from Limerick?
(ii) Halfway from Wexford to Limerick (at point A), one of the swallows feels ill and decides to head home to Killarney, how far a journey will he have?

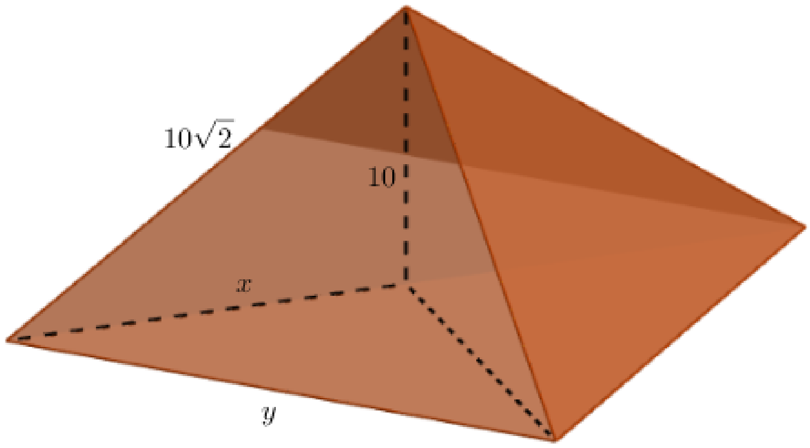


3. A space station in Peru can collect the following measurements (where Mm is 1000km).
(i) Using the Cosine rule, show that the distance x must satisfy the quadratic equation $3x^2 - 2x - 4 = 0$.
(ii) Solve this quadratic to the nearest two decimal places, explaining which answer is correct.

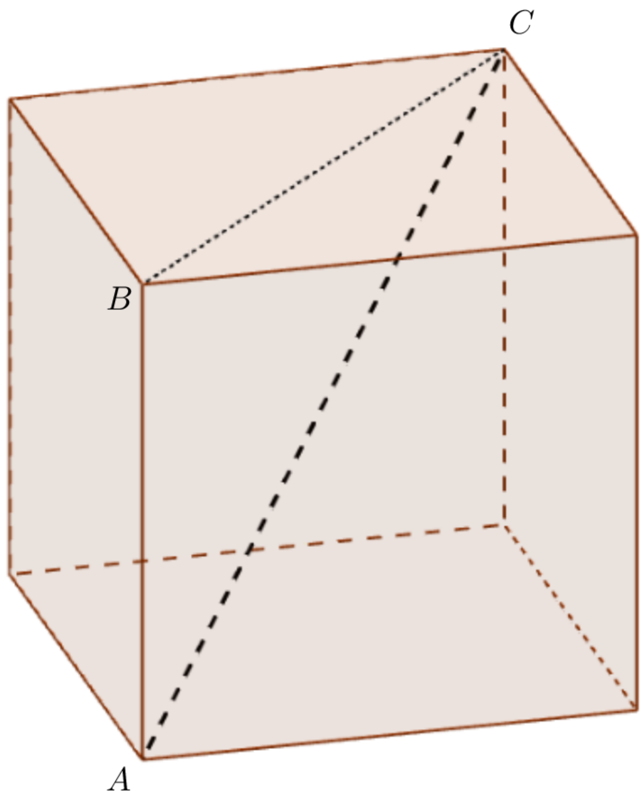


6.3 3D Problems

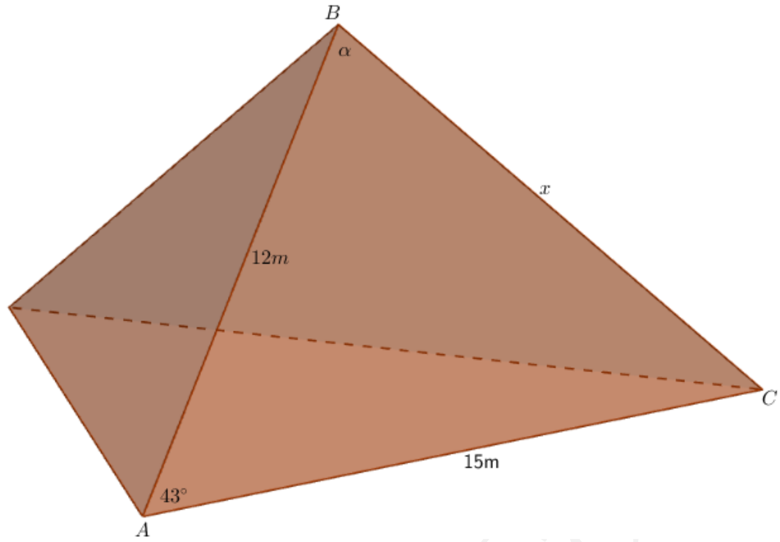
1. The following is a pyramid of vertical height 10m and slant height $10\sqrt{2}$m. Using the relevant trigonometric formulae, calculate the unknown lengths x and y. (As with all pyramids, you can assume all base widths are the same and the shape is symmetrical.)



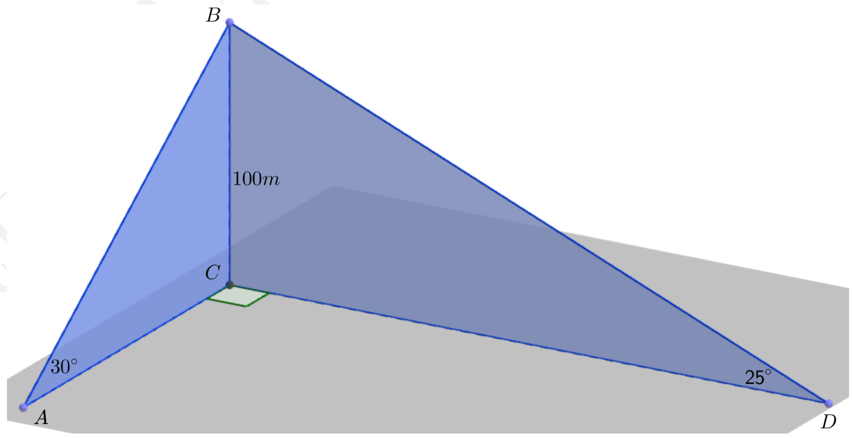
2. The following is a cube with sides of width 1m. By first calculating the length of the diagonal on the top face of the cube, $|BC|$, calculate the length of the diagonal through the cube $|AC|$.



3. The following shape is an irregular tetrahedron. On the face $\triangle ABC$, the base length is 15m and one of the slant heights is 12m. Given the angle in between to be 43°, calculate
   (i) The length of the unknown side x.
   (ii) The angle α at the apex of the tetrahedron.

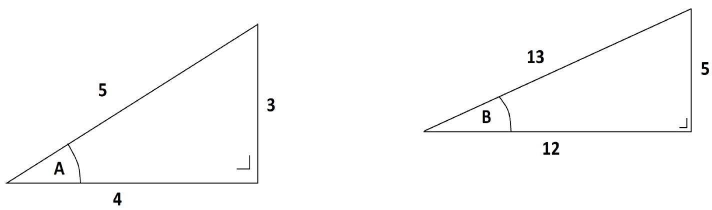


4. From two points on the ground, A and D, a bird in the sky can be seen at the point B. If the bird is flying 100m above the ground, and the angles of elevation measured are given in the diagram, calculate
   (i) The distance $|AB|$.
   (ii) The distance $|DB|$.
   (iii) The distance $|AD|$.



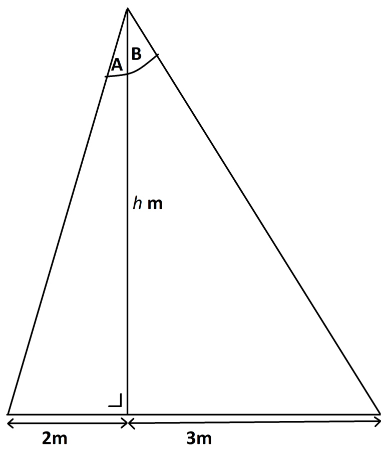
6.4 Trigonometric Identities

1. Prove: $\sin x \sec x = \tan x$
2. Prove: $\dfrac{\tan x \cos x}{\sin x} = 1$
3. Prove: $\dfrac{\sec x \csc x}{\csc^2 x} = \tan x$
4. Prove: $\dfrac{\sin x + \cos x}{\cos x} = \tan x + 1$
5. Prove: $\cos(A+B)\cos B + \sin(A+B)\sin B = \cos A$
6. Prove: $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
7. Prove: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
8. Prove: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
9. Prove: $1 - \cos^2 x \tan^2 x = \cos^2 x$
10. Prove: $\sec^2 x = 1 + \tan^2 x$
11. Prove: $\dfrac{\tan x}{1 + \tan x} = \dfrac{\sin x}{\cos x + \sin x}$
12. Prove: $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$



13. Use the triangles shown above to find the value of
    (a) $\sin(A+B)$
    (b) $\cos(A-B)$

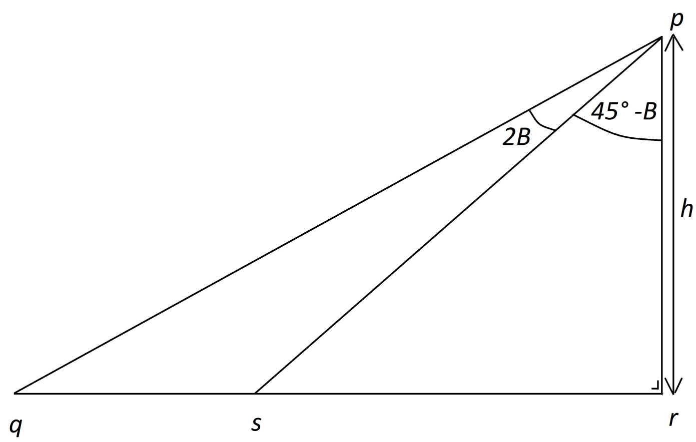
14. The diagram shows a triangle of height h m. The angles A and B are such that $A + B = 45°$. By using the expansion of $\tan(A + B)$, or otherwise, find the value of h



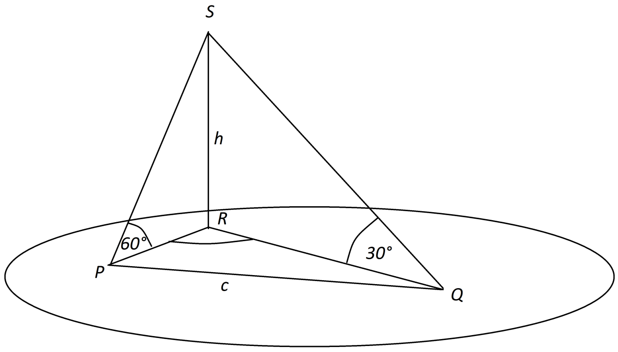
15. In the triangle pqr, $| < qrp| = 90°$ and $|rp| = h$. s is a point on $[qr]$ such that $| < spq| = 2B$ and $| < rps| = 45° - B$, $0° < B < 45°$.

    i. Show that $|sr| = h \tan(45 - B)$

    ii. Hence, or otherwise, show that $|qs| = 2h \tan 2B$

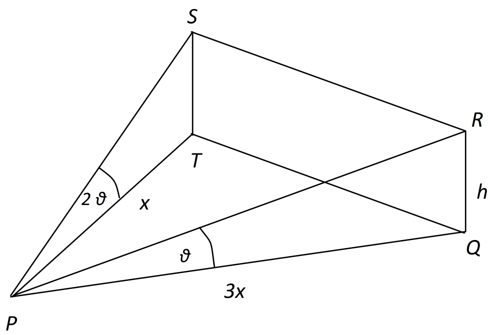


16. P, Q and R are three points on the horizontal ground. $[SR]$ is a vertical pole of height h metres.
The angle of elevation of S from P is $60°$, and the angle of elevation of S from Q IS



$30°$.
Given that $3c^2 = 13h^2$, find $|<PRQ|$.

17. $QRST$ is a vertical rectangular wall of height h on level ground. P is a point on the ground in front of the wall. The angle of elevation of R from P is θ, and the angle of



elevation of S from P is 2θ. $|PQ| = 3|PT|$.
Find the measure of θ.

6.5 Trigonometric Equations

1. Solve each of the following trigonometric equations for θ, where $0° \leq \theta \leq 360°$:
   i. $\sin\theta = 0.5$
   ii. $\cos\theta = \frac{\sqrt{3}}{2}$
   iii. $\tan\theta = 1$
   iv. $\cos\theta = -\frac{\sqrt{3}}{2}$

v. $\tan\theta = -\sqrt{3}$

vi. $2\sin\theta = \sqrt{3}$

vii. $2\cos\theta = -1$

viii. $2\sin^2\theta = 1$

ix. $4\cos^2\theta = 3$

2. Solve each of the following trigonometric equations for x, where $0° \leq x \leq 360°$:

i. $\sin 2x = \frac{1}{2}$

ii. $\tan 3x = -\frac{1}{\sqrt{3}}$

iii. $\cos 2x = \frac{1}{\sqrt{2}}$

iv. $\sin 2x = -\frac{1}{\sqrt{2}}$

v. $\sin 3x = 0$

vi. $\cos 4x + 1 = 0$

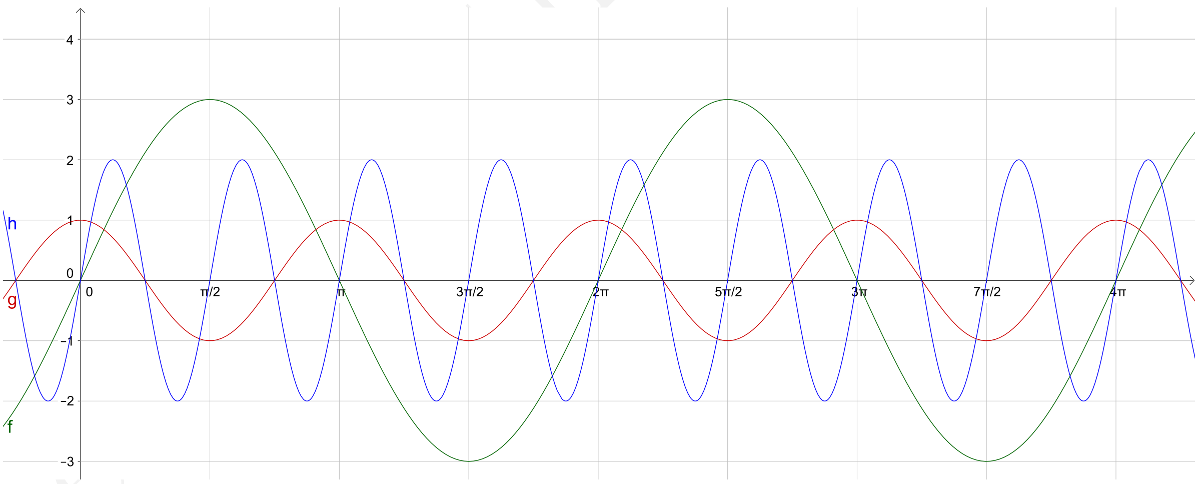
6.6 Trigonometric Functions

1. (a) What is the range and the period of the function $f(x) = 3\sin x$?

(b) What is the period and the range of the function $g(x) = \cos(2x)$?

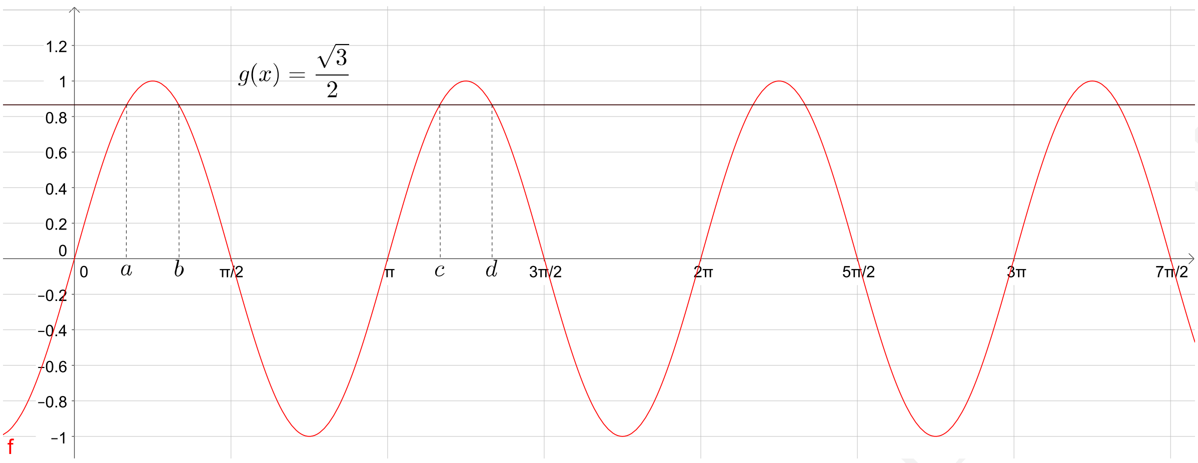
(c) Label $f(x)$ and $g(x)$ on the graph below.

(d) $h(x)$ is also on the graph below. What is the function $h(x)$?



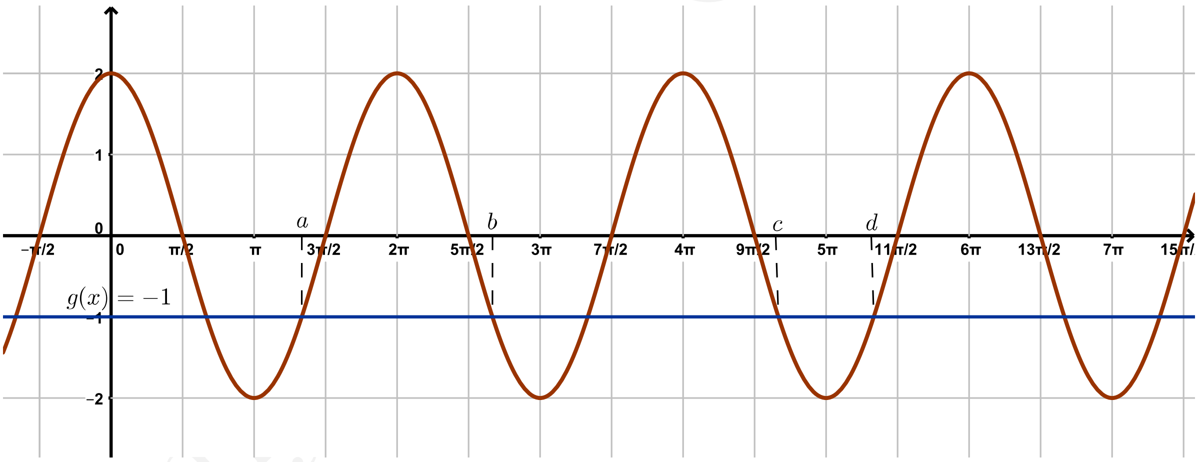
2. (a) Identify $f(x)$, the trigonometric function in the graph below.

   (b) $g(x) = \frac{\sqrt{3}}{2}$. By letting $f(x) = g(x)$ and solving the resulting equation, find the values of a, b, c and d on the graph below.



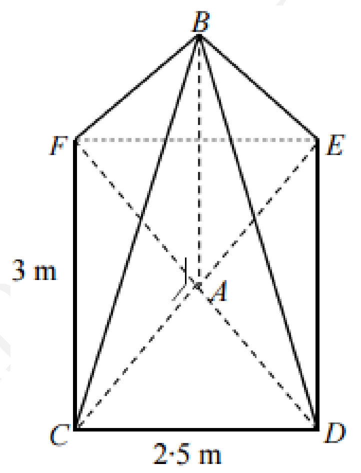
3. (a) Identify $f(x)$, the trigonometric function in the diagram below.

   (b) $g(x) = -1$. By letting $f(x) = g(x)$ and solving the resulting equation, find the values of a, b, c and d.

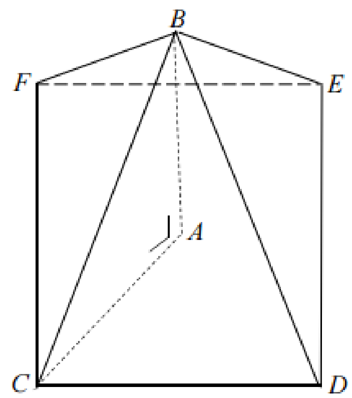


6.7 Exam Questions

1. **2016 Paper 2** A glass Roof Lantern in the shape of a pyramid has a rectangular base $CDEF$ and its apex is at B as shown. The vertical height of the pyramid is $|AB|$, where A is the point of intersection of the diagonals of the base as shown in the diagram.
   Also $|CD| = 25$ m and $|CF| = 3m$.
   (a) i. Show that $|AC| = 1.95$ m, correct to two decimal places.
   ii. The angle of elevation of B from C is $50°$ (i.e. $|BCA| = 50°$).
   Show that $|AB| = 2.3$ m, correct to one decimal place.
   iii. Find $|BC|$, correct to the nearest metre.
   iv. Find $|BCD|$, correct to the nearest degree.
   v. Find the area of glass required to glaze all four triangular sides of the pyramid.
   Give your answer correct to the nearest m^2.



   (b) Another Roof Lantern, in the shape of a pyramid, has a square base $CDEF$. The vertical height $|AB| = 3$ m, where A is the point of intersection of the diagonals of the base as shown.
   The angle of elevation of B from C is $60°$ (i.e. $|BCA| = 60°$). Find the length of the side of the square base of the lantern.
   Give your answer in the form $\sqrt{a}$ m, where $a \in \mathbb{N}$.



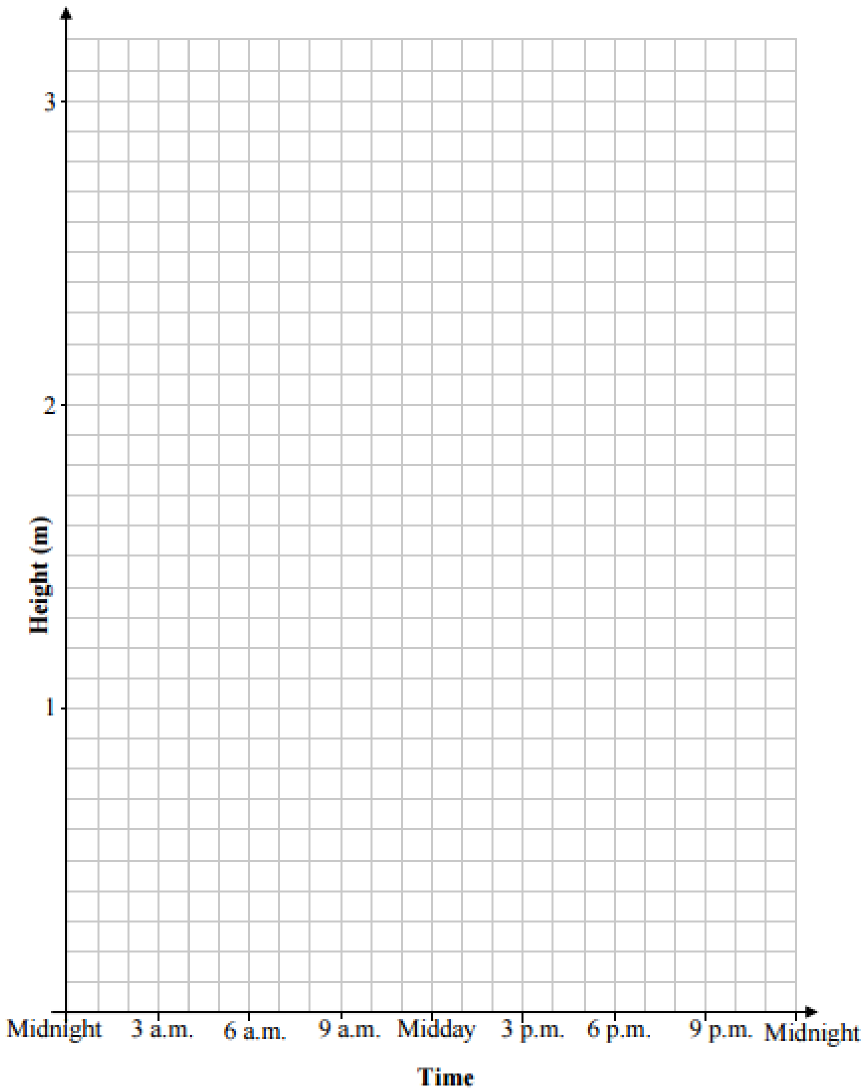
2. The height of the water in a port was measured over a period of time. The average height was found to be 1.6 m. The height measured in metres, $h(t)$, was modelled using the function

$$h(t) = 1.6 + 1.5\cos\left(\frac{\pi}{6}t\right)$$

   where t represents the number of hours since the last recorded high tide and $(\frac{\pi}{6}t)$ is expressed in radians.
   (a) Find the period and range of $h(t)$.
   (b) Find the maximum height of water in the port.
   (c) Find the rate at which the height of the water is changing when $t = 2$, correct to two decimal places. Explain your answer in the context of the question.
   (d) i. On a particular day the high tide occurred at midnight (i.e. $t = 0$). Use the function to complete the table and show the height, $h(t)$, of the water between midnight and the following midnight.

$h(t) = 1{\cdot}6 + 1{\cdot}5\cos\left(\frac{\pi}{6}t\right)$									
Time	Midnight	3 a.m.	6 a.m.	9 a.m.	12 noon	3 p.m.	6 p.m.	9 p.m.	Midnight
t (hours)	0	3							
$h(t)$ (m)									

(ii) Sketch the graph of $h(t)$ between midnight and the following midnight.



ii. Find, from your sketch, the difference in water height between low and high tide.

iii. A fully loaded barge enters the port, unloads its cargo and departs some time later. The fully loaded barge requires a minimum water level of 2 m. When the barge is unloaded it only requires 1.5 m. Use your graph to estimate the maximum amount of time that the barge can spend in port, without resting on the sea-bed.

3. Show that $\dfrac{\cos 7A + \cos A}{\sin 7A - \sin A} = \cot 3A$

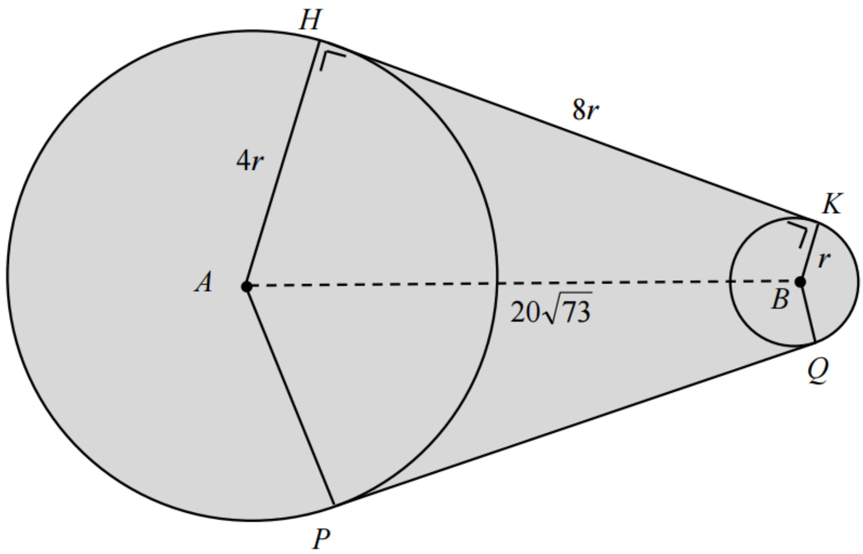
4. Given that $\cos 2\theta = \dfrac{1}{9}$, find $\cos\theta$ in the form $\pm\frac{\sqrt{a}}{b}$, where $a, b \in \mathbb{N}$

2015 Paper 2

5. (a) Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

   (b) Find all the values of x for which $\sin(3x) = \dfrac{\sqrt{3}}{2}$, $0 \leq x \leq 360$, x in degrees.

6. A flat machine consists of two circular ends attached to a plate, as shown (diagram not to scale).
The sides of the plate, HK and PQ, are tangential to each circle.
The larger circle has a centre A and a radius $4r$ cm.
The smaller circle has centre B and radius r cm.
The length of $[HK]$ is $8r$ cm and $|AB| = 20\sqrt{73}$ cm.



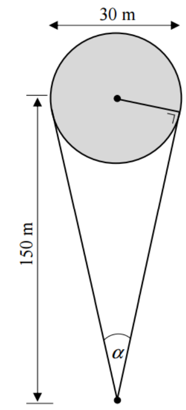
   (a) Find r, the radius of the smaller circle. (Hint: Draw $BT \parallel KH$, $T \in AH$.)

   (b) Find the area of the quadrilateral $ABKH$.

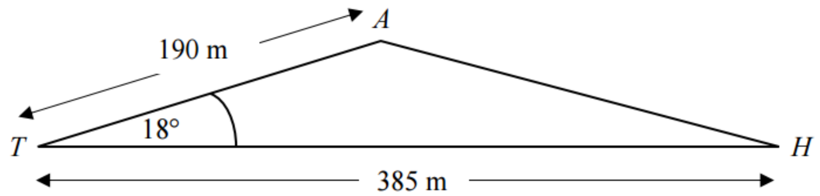
   (c) i. Find $|\angle HAP|$, in degrees, correct to one decimal place.

   ii. Find the area of the machine part, correct to the nearest cm^2.

7. (a) Joan is playing golf. She is 150 m from the centre of a circular green of diameter 30 m. The diagram shows the range of directions in which Joan can hit the ball so that is could land on the green. Find α, the measure of the angle of this range of directions. Give your answer, in degrees, correct to one decimal place.



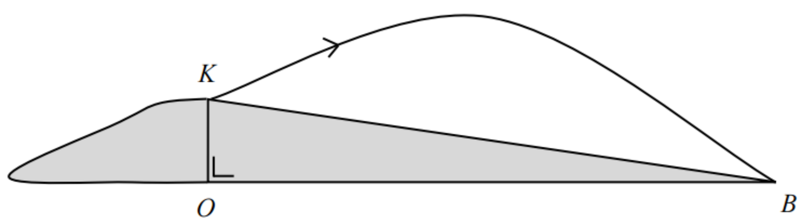
(b) At the next hole, Joan, at T, attempts to hit the ball in the direction of the hole H. Her shot is off target and the ball lands at A, a distance of 190 metres from T, where $|\angle ATH| = 18°$. $|TH|$ is 385 metres. Find $|AH|$, the distance from the ball to the hole, correct to the nearest metre.



(c) At another hole, where the ground is not level, Joan hits the ball from K, as shown. The ball lands at B. The height of the ball, in metres, above the horizontal line OB is given by:

$$h = -6t^2 + 22t + 8$$

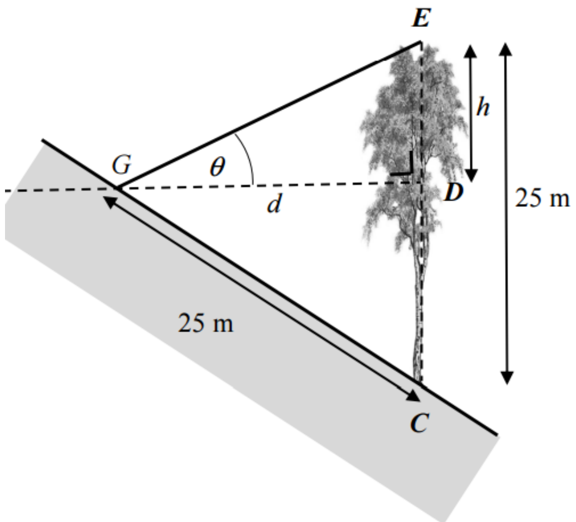
where t is the time in seconds after the ball is struck and h is the height of the ball.



i. Find the height of K above OB.

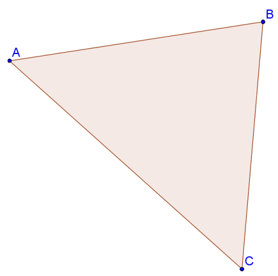
ii. The horizontal speed of the ball over the straight distance $[OB]$ is a constant 38 m s^{-1}.
Find the angle of elevation of K from B, correct to the nearest degree.

(d) At a later hole, Joan's first shot lands at the point G, on ground that is sloping downwards, as shown. A vertical tree, $[CE]$, 25 metres high, stands between G and the hole. The distance, $|GC|$, from the ball to the bottom of the tree is also 25 metres.
The angle of elevation at G to the top of the tree, E, is θ, where $\theta = \tan^{-1}\frac{1}{2}$.
The height of the top of the tree above the horizontal, GD, is h metres and $|GD| = d$ metres.
i. Write d and $|CD|$ in terms of h.
ii. Hence, or otherwise, find h.



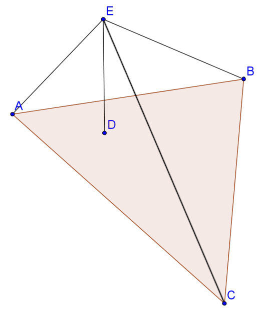
2014 Paper 2

8. The length of the sides of a flat triangular field ACB are, $|AB| = 120m$, $|BC| = 134m$ and $|AC| = 150m$.



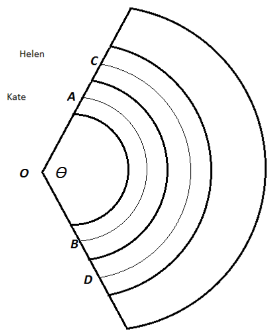
(a) i. Find $|\angle CBA|$. Give your answer, in degrees, correct to two decimal places.
ii. Find the area of the triangle ACB correct to the nearest whole number.

(b) A vertical mast, $[DE]$, is fixed at the circumcentre, D, of the triangle. The mast is held in place by three taut cables $[EA]$, $[EB]$ and $[EC]$. Explain why the three cables are equal in length. (See figure on next page)



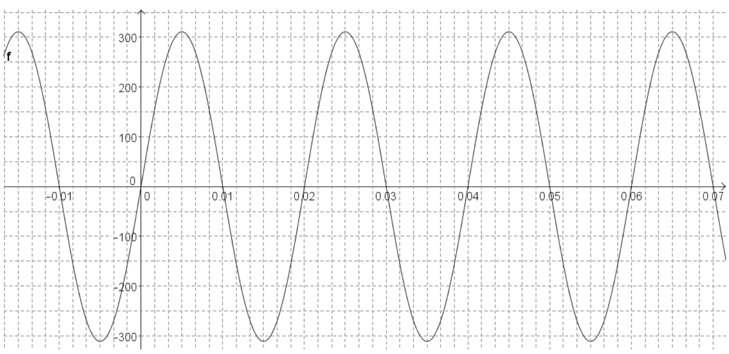
9. (a) Prove that $\cos 2A = \cos^2 A - sin^2 A$.

   (b) The diagram shows part of the circular end of a running track with three running lanes shown. The centre of each of the circular boundaries of the lanes is at O.



- Kate runs in the middle of lane 1, form A to B as shown.
- Helen runs in the middle of lane 2, from C to D as shown.
- Helen runs 3 m further than Kate.
- $|<AOB| = |<COD| = \theta$ radians.
- If each lane is 1.2m wide, find θ .

10. The graph below shows the voltage, V, in an electric circuit as a function of time, t. The voltage is given by the formula $V = 311\sin(100\pi t)$, where V is in volts and t is in seconds.



   (a) i. Write down the range of the function.

ii. How many complete periods are there in one second?

(b) i. The table below gives the voltage,correct to the nearest whole number, at equally spaced intervals from t_1 to t_12 over one complete period. Use the entries given in the table and the properties of the function to complete the table

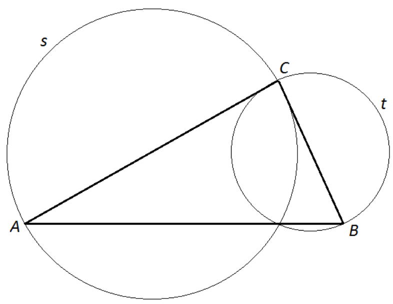
t	t_1	t_2	t_3	t_4	t_5	$t_6 = 0.01$	t_7	t_8	t_9	t_10	t_11	$t_12 = 0.02$
V	156	269	311									

ii. Using a calculator, or otherwise, calculate the standard deviation, σ of the twelve values of V in the table, correct to the nearest whole number.

(c) i. The standard deviation, σ, of closely spaced values of any function of the form $V = a\sin(bt)$, over 1 full period, is given by $k\sigma = V_max$, where k is a constant that does not depend on a or b, and V_max is the maximum value of the function. Use the function $V = 311\sin(100\pi t)$ to find an approximate value for k correct to three decimal places.

ii. Using your answer from part **(c)(i)**, or otherwise, find the value of b required so that the function $V = a\sin(bt)$ has 60 complete periods in one second and the approximate value of a so that it has a standard deviation of 110 volts.

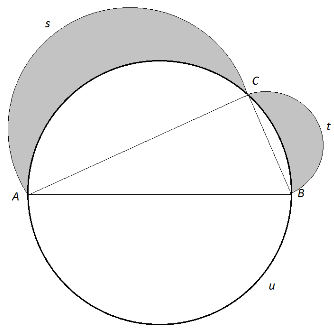
11. The triangle ABC is right-angled at C. The circle s has diameter $[AC]$ and the circle t has a diameter $[CB]$.



i. Draw the circle u which has a diameter $[AB]$.

ii. Prove that in any right-angles triangle ABC, the area of the circle u equals the sum of the areas of the circles s and t.

iii. The diagram shows the right-angled triangle ABC and arcs of the circles s, t and u. Each of the shaded areas in the diagram is called a lune, a crescent-shaped area bounded by arcs of the circles. Prove that the sum of the areas of the two shaded lunes is equal to the area of the triangle ABC.



2013 Paper 2

12. (a) In a triangle ABC, the lengths of the sides are a, b and c. Using a formula for the area of a triangle, or otherwise, prove that $\frac{a}{\sin <A} = \frac{b}{\sin <B} = \frac{c}{\sin <C}$.

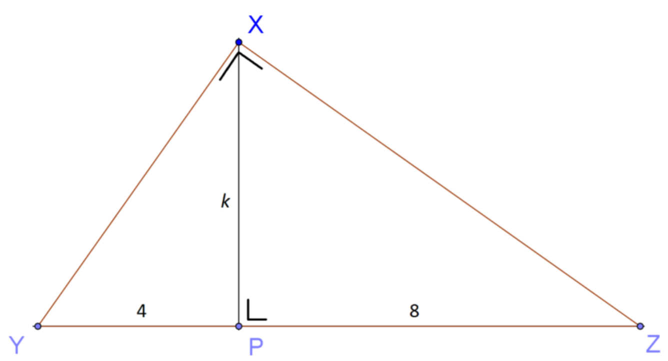
(b) In a triangle XYZ, $|XY| = 5cm$, $|XZ| = 3cm$ and $|<XYZ| = 27°$

i. Find the two possible values of $|<XYZ|$. Give your answer correct to the nearest degree.

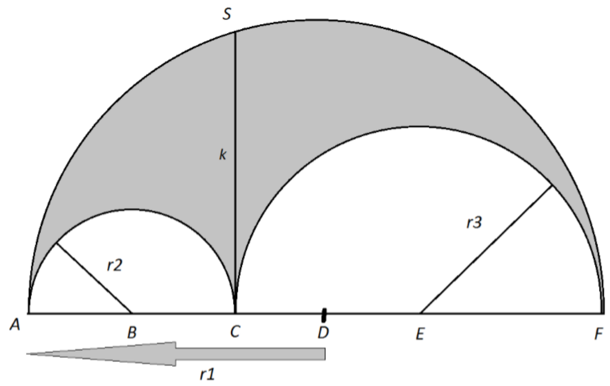
ii. Draw a sketch of the triangle XYZ, showing the two possible positions of the point Z.

(c) In the case that $|<XYZ|<90°$, write down $|<ZXY|$, and hence find the area of the triangle XYZ to the nearest integer.

13. (a) The triangle XYZ is right-angled at X and XP is perpendicular to YZ. $|YP| = 4$, $|PZ| = 8$ and $|PX| = k$. Find the value of k.



(b) The shaded region in the diagram below is called an arbelos. It is a plane semicircular region of radius r_1 from which semicircles of radius r_2 and r_3 are removed, as shown. In the diagram $SC \perp AF$ and $|SC| = k$



i. Show that, for fixed r_1, the perimeter of the arbelos is independent of the values of r_2 and r_3.

ii. If $r_2 = 2$ and $r_3 = 4$, show that the area of the arbelos is the same as the area of the circle of diameter k.

(c) To investigate the area of an arbelos, a student fixed the value of r_1 at $6cm$ and completed the following table for different values of r_2 and r_3.

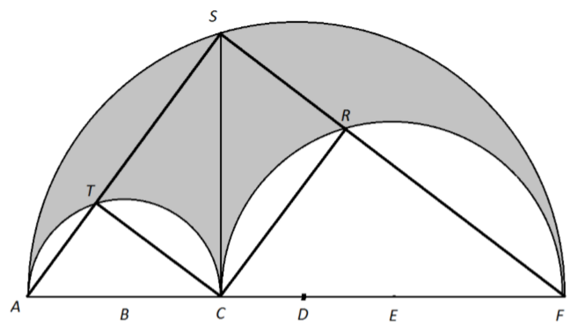
i. Complete the table.

r_1	r_2	r_3	Area of arbelos
6	1		
6	2		
6	3		
6	4		
6	5		

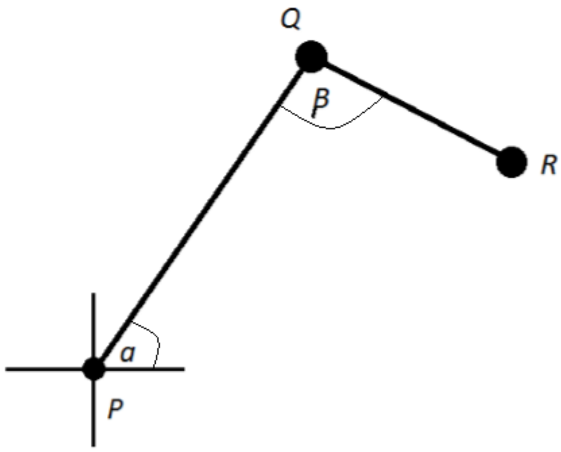
ii. In general, for $r_1 = 6$cm and $r_2 = x$, $0<x<6$, $x \in \mathbb{R}$ find an expression in x for the area of the arbelos.

iii. Hence, or otherwise, find the maximum area of the arbelos that can be formed in a semicircle of radius $6cm$.

(d) AS and FS cut the two smaller semicircle at T and R respectively. Prove that $RSTC$ is a rectangle.



14. **2012 Paper 2** The diagram is a representation of a robotic arm that can move in a vertical plane. The point P is fixed, and so are the lengths of the two segments of the arm. The controller can vary the angles α and β from $0°$ to $180°$.

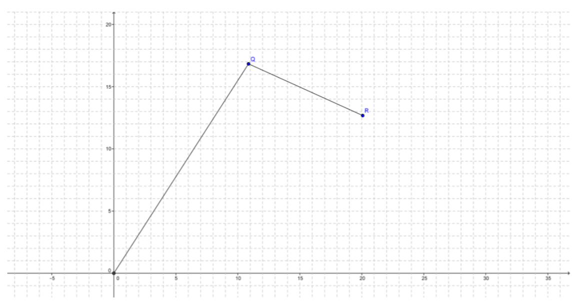


(a) Given that $|PQ| = 20cm$ and $|QR| = 12cm$, determine the values of the angles α and β so as to locate R, the tip of the arm, at point that is $24cm$ to the right of P, and $7cm$ higher than P. Give your answer correct to the nearest degree.

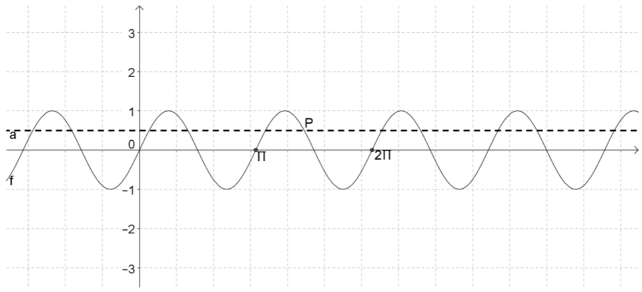
(b) In setting the arm to the position described in part **a**, which will cause the greater error in the location of R: an error of $1°$ in the value of α or an error of $1°$ in the value of β? Justify your answer. You must assume that if a point moves along a circle through a small angle, then it's distance from its starting point is equal to the length of the arc travelled.

(c) The answer to part **b** above depends on the particular position of the arm. That is, in certain positions, the location of R is more sensitive to small errors in α than to small errors in β, while in other positions, the reverse is true. Describe, with justification, the conditions under which each of these two situations arise.

(d) Illustrate the set of all possible locations of the point R on the co-ordinate diagram below. Take P as the origin and take each unit in the diagram to represent a centimetre in reality. Note that α and β can vary only from $0°$ to $180°$.



15. **2014 Sample Paper 2** The diagram below shows the graph of the function $f : x| \rightarrow \sin 2x$. The line $2y = 1$ is also shown.



(a) On the same diagram above, sketch the graphs of $g : x \rightarrow \sin x$ and $h : x \rightarrow 3\sin 2x$. Indicate clearly which is g and which is h.

(b) Find the co-ordinates of point P in the diagram.

16. **2011 Paper 2** Scientists use information about seismic waves from earthquakes to find out about the internal structure of the earth. The diagram represents a circular cross section of the earth. The dashed curve represents the path of a seismic wave travelling through the earth from an earthquake near the surface at A to a monitoring station at B. The radius of the earth is 6.4 units and the path of the wave is a circular arc of radius 29.1 units, where 1 unit = 1000 km. Based on information from other stations, it is known that this particular path just touches the earth's core. The angle AOB measures 104°, where O is the centre of the earth. Find the radius of the earth's core.