5 Calculus 2 - Integration

5.1 Indefinite Integration

1. $\int(x^3+x^2+x+1)\mathrm{d}x$
2. $\int(5x^3+3x^2-2x-7)\mathrm{d}x$
3. $\int x^{-3}\,\mathrm{d}x$
4. $\int 3x^{-2}\,\mathrm{d}x$
5. $\int \frac{1}{x^2}\,\mathrm{d}x$
6. $\int \sqrt{x}\,\mathrm{d}x$
7. $\int \frac{1}{\sqrt{x}}\,\mathrm{d}x$
8. $\int \sqrt[3]{x}\,\mathrm{d}x$
9. $\int \dfrac{x^3+3x^2-2x}{x}\,\mathrm{d}x$
10. $\int \dfrac{5x^3+3x^2+3}{x^2}\,\mathrm{d}x$
11. $\int(x^2+5)^2\,\mathrm{d}x$
12. $\int \sqrt{x}(x-5)\,\mathrm{d}x$
13. If $f'(x)=2x^3+7$ and $f(2)=25$, find$f(x)$
14. If $f'(x)=2x+\sqrt{x}$ and $f(1)=\frac{8}{3}$, find $f(x)$
15. A curve contains the point $(1,10)$ and the slope at any point is given by $\frac{dy}{dx}=2x+5$. Find the equation of the curve.
16. A curve contains the point $(3,0)$ and the slope of the curve at any point is given by $f'(x)=3x^2-10x+7$. Find $f(x)$.

5.2 Definite Integration

1. $\int_1^2 (2x+4)\ \mathrm{d}x$
2. $\int_{-1}^4 (3x^2+8x-1)\ \mathrm{d}x$
3. $\int_0^2 x^2(x-1)\ \mathrm{d}x$
4. $\int_1^2 \frac{1}{x^2}\ \mathrm{d}x$
5. $\int_1^2 \dfrac{x^3+3x^2-2x}{x}\ \mathrm{d}x$
6. $\int_4^9 (\sqrt{x}+\frac{1}{\sqrt{x}})\ \mathrm{d}x$

5.3 Exponential Functions

1. $\int e^{5x}\ \mathrm{d}x$
2. $\int e^{2x+1}\ \mathrm{d}x$
3. $\int_0^2 e^{2-3x}\ \mathrm{d}x$
4. $\int_0^3 (e^{2x}+\frac{1}{e^{2x}})\ \mathrm{d}x$
5. $\int 3^x\ \mathrm{d}x$
6. $\int_0^2 4^x\ \mathrm{d}x$
7. $\int_0^4 3^{x-4}\ \mathrm{d}x$
8. If $f(x)=3xe^x$, find $f'(x)$.
   Hence, calculate $\int 3xe^x\ \mathrm{d}x$.
9. If $f(x)=2xe^{5x}$, find $f'(x)$.
   Hence, calculate $\int 10xe^{5x}\ \mathrm{d}x$.
10. Let $f'(x)=ke^{2x}-3$, where $k\in R$. Find $f(x)$, given that the slope of the tangent to $f(x)$ at the point $(0,3)$ is 1.
11. Let $f'(x)=ke^{-3x}+7$, where $k\in R$. Find $f(x)$, given that the slope of the tangent to $f(x)$ at the point $(0,2)$ is 10.

5.4 Trigonometric Integration

1. $\int \sin 3x\ \mathrm{d}x$
2. $\int_{\frac{\pi}{2}}^{\pi} \cos 4x\ \mathrm{d}x$
3. $\int (\sin 3\theta - 2\cos 5\theta)\ \mathrm{d}\theta$
4. $\int 2\cos 4x \sin x\ \mathrm{d}x$

5. $\int \sin 3x \cos 2x \, \mathrm{d}x$

6. $\int_0^{\frac{\pi}{3}} \cos 3x \cos 4x \, \mathrm{d}x$

7. $\int_0^{\frac{\pi}{2}} \sin 2x \cos x \, \mathrm{d}x$

8. If $f(x) = 3x \sin x$, find $f'(x)$.
   Hence, calculate $\int 3x \cos x \, \mathrm{d}x$.

9. If $f(x) = 2x \cos 3x$, find $f'(x)$.
   Hence, calculate $\int 6x \sin 3x \, \mathrm{d}x$.

5.5 Average Value

1. Find the average value of $f(x) = 3x^2$ on $1 \leq x \leq 4$.

2. Find the average value of $f(x) = \sqrt{x} + 8x$ on $4 \leq x \leq 9$.

3. Find the average value of $f(x) = \sin x$ on $0 \leq x \leq \frac{\pi}{2}$.

4. Find the average value of $f(x) = 3 \cos 2x$ on $0 \leq x \leq \pi$.

5. Find the average value of $f(x) = x^3 + 3x^2 - 7x$ on $-2 \leq x \leq 3$.

6. Find the average value of $f(x) = 6e^{3x}$ on $0 \leq x \leq 1$.

7. **(2015 Paper 1)**
   The approximate length of the day in Galway, measured in hours from sunrise to sunset, may be calculated using the function

$$f(t) = 12.25 + 4.75 \sin\left(\frac{2\pi}{365}t\right),$$

   where t is the number of days after March 21^{st} and $\left(\frac{2\pi}{365}t\right)$ is expressed in radians.
   Use integration to find the average length of the day in Galway over the six months form March 21^{st} to September 21^{st} (184 days). Give you answer in hours and minutes, correct to the nearest minute.

8. **(2016 Leamy Maths Pre Paper 1)**
   The amount of milk produced by each cow varies with time. The number of litres of milk may be approximated using the function

$$f(t) = 20 - 4 \cos\left(\frac{2\pi}{366}t\right),$$

   where t is the number of days from (and including) the rst of January and the ratio $\frac{2\pi}{366}$ is expressed in radians. Note this is a leap year.

   How many litres of milk does a cow produce on average between 1 January and 31 March?(91 days)

5.6 Rates of Change

1. The distance, in metres, travelled by an object in t seconds is given by the formula
$$s = t^3 - 3t^2 + 5t.$$
   (a) Calculate the initial position and the initial velocity.
   (b) Calculate the velocity at 5 seconds.
   (c) Calculate the acceleration at 3 seconds.

2. An object is projected upwards and its height in metres above the ground after t seconds is
$$h = 490t - 4.9t^2.$$
   i. After how many seconds is the object momentarily at rest?
   ii. Find the maximum height the object reaches, to the nearest km.

3. A particle moving in a straight line is s cm from the point O at time t seconds ($t \geq 0$), where $s = t^3 - 9t^2 + 24t$.
   i. Find its initial position and velocity.
   ii. At what times is the particle stationary?
   iii. What is the position of the particle when it is stationary?
   iv. For how long is the particle's velocity negative?
   v. Find its acceleration at any time.
   vi. When is the particle's acceleration zero, and what is its velocity and position at that time?

4. An object is projected upwards from the ground with an initial speed of 9.8 m/s. Its height in metres after t seconds is given by $y = 9.8t - 4.9t^2$
   i. What is the acceleration of the ball at any time t?
   ii. How high does the ball go?
   iii. How fast is it moving when it strikes the ground?

5. A particle starts from a fixed point O and moves in a straight line. The velocity (v) of the particle, at any time (t) seconds, is given by $v = 2t + 1$ m/s.
   (a) What is the velocity after 5 seconds?
   (b) What is the acceleration of the particle at any time?
   (c) What is the position (s) of the particle at any time, considering that $s = 0$ when $t = 0$?
   (d) What is the position of the particle after 4 seconds?

6. The acceleration of an object is given by $a = 2t - 10$ m/s^2.
   (a) Find the velocity, $v(t)$, at any time, given that the initial velocity is 25 m/s.
   (b) Find the velocity at 3 seconds. What is the acceleration at this time?

(c) Find the position, $s(t)$, at any time, given that the initial position is at 2m.

(d) What is the position of the object when it is stationary?

7. Raindrops grow as they fall, their surface area increases, and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and it's downward acceleration is given by;

$$a(t) = \begin{cases} 9 - 0.9t, & : 0 \leq t < 10 \\ 0, & t \geq 10 \end{cases}$$

The raindrop is initially 500m above the ground.

(a) Find the velocity function $v(t)$ of the raindrop after t seconds, $0 \leq t \leq 10$.

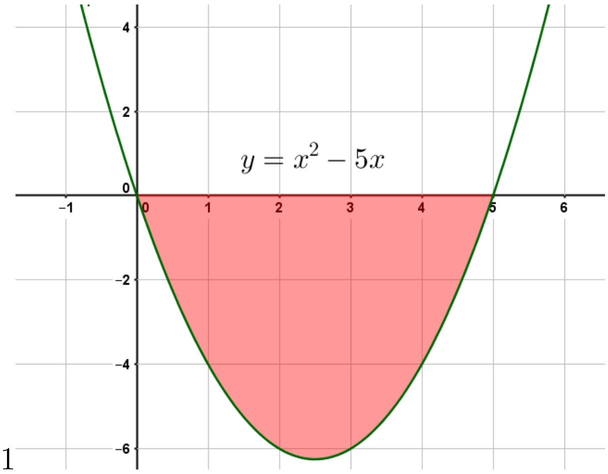
(b) Find the velocity at 10 seconds. Hence at what speed does the raindrop hit the ground?

(c) Find $s(t)$, the distance travelled by the raindrop in t seconds, $0 \leq t \leq 10$.

(d) How long does it take the raindrop to fall?

8. A rock falls from the top of a massive cliff and hits the ground at the base of the cliff at a speed of 147 m/s. How high is the cliff? (Hint: Acceleration due to gravity = -9.8m/s^2. Represent all velocities and accelerations as negative values, since the rock is falling towards the ground. Ignore air resistance)

5.7 Finding Areas by Integration - Calculate the Shaded Area



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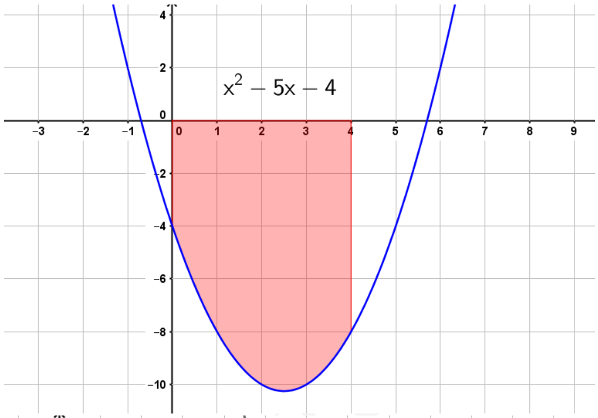

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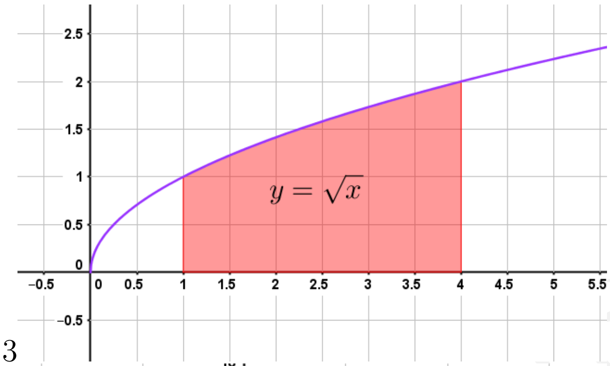

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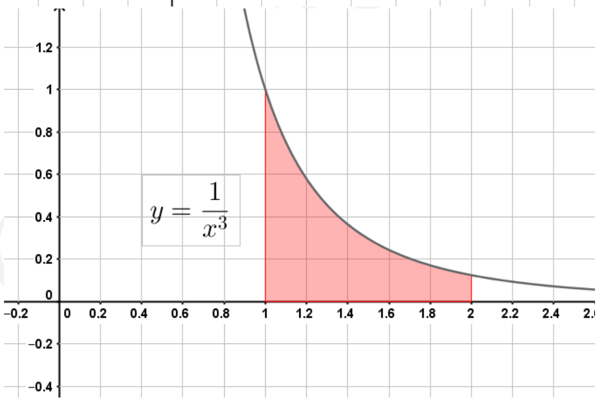

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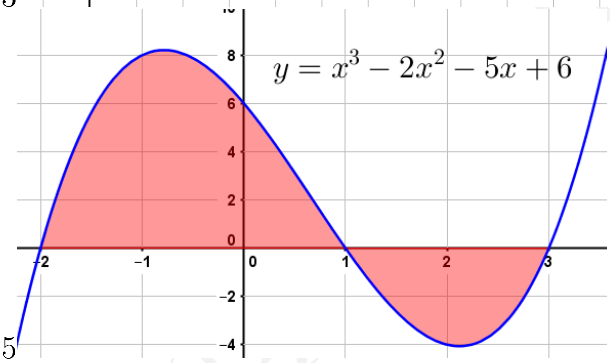

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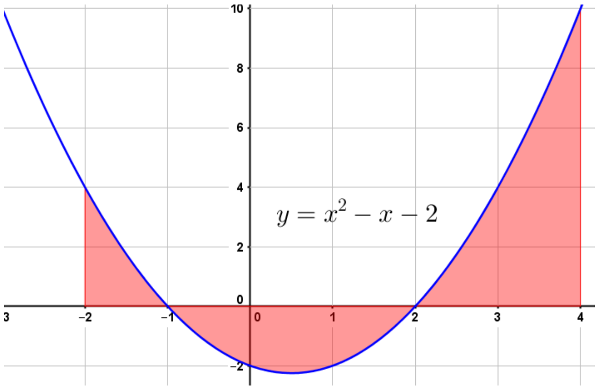

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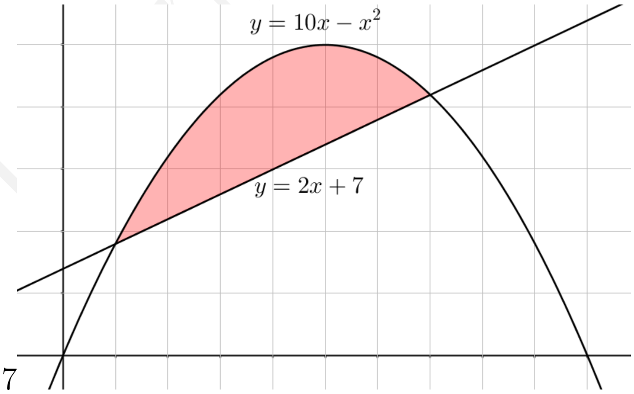

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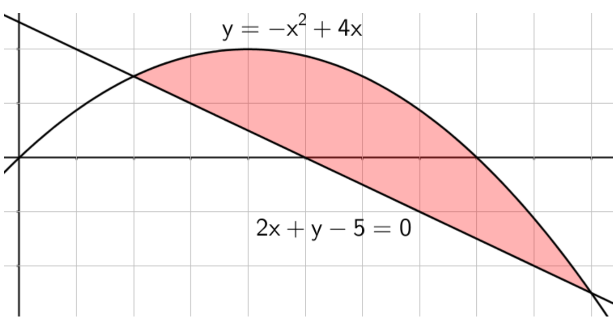

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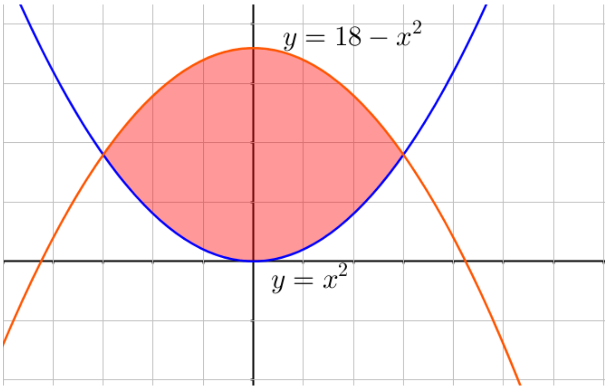

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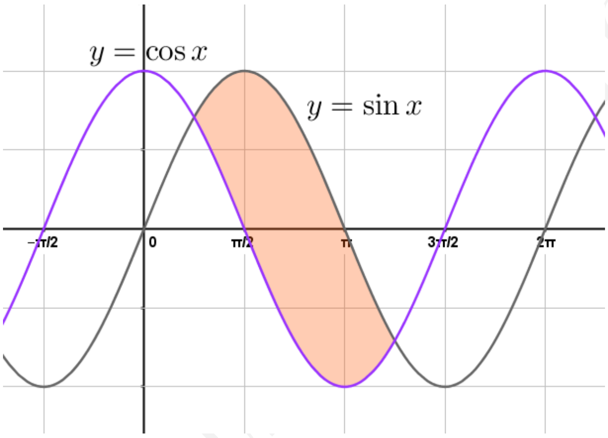

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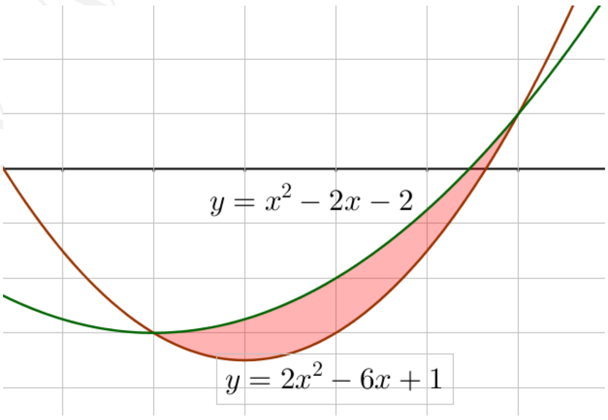


5.8 Exam Questions

1. **2016 P1** An inflated ball is kicked into the air from a point O on the ground. Taking O as the origin, $(x, f(x))$ approximately describes the path followed by the ball in the air, where

$$f(x) = -x^2 + 10x$$

and both x and $f(x)$ are measured in metres.

   i. Find the values of x when the ball is on the ground.

   ii. Find the average height of the ball above the ground, during the interval from when it is kicked until it hits the ground again.

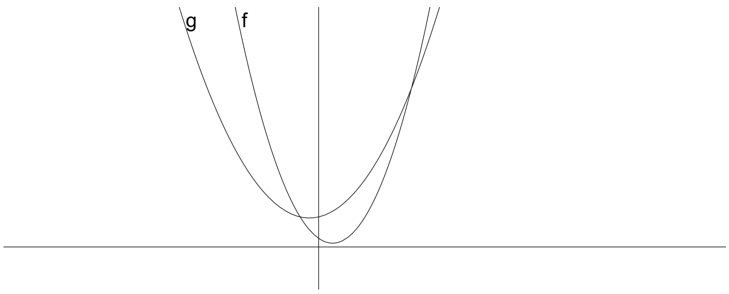
2. **2014 Paper 1**
   (a) Find $\int 5\cos 3x dx$.
   (b) The slope of a tangent to a curve $y = f(x)$ at each point (x, y) is $2x - 2$. The curve cuts the $x - axis$ at $(-2, 0)$.
      i. Find the equation of $f(x)$.
      ii. Find the average value of f over the interval $0 \leq x \leq 3$, $x \in \mathbb{R}$

3. **2013 Paper 1**
The speed at which a raindrop falls increases until a maximum speed, called its terminal velocity, is reached. The raindrop then continues to fall at this terminal velocity. The distance , s metres, it falls is given by

$$s(t) = \begin{cases} 6t + 0.3t^2 - 0.01t^3, & : 0 \leq t \leq 10 \\ k(t - 10), & t > 10 \end{cases}$$

where t is the time in seconds from the instant the raindrop begins to fall and k is a constant.
   (a) How far this raindrop fallen after 10 seconds?
   (b) After how many seconds is the raindrop falling at a speed of 8.25 metres per second?
   (c) The acceleration of the raindrop is decreasing for the first 10 seconds of its fall. Find the value of t for which the acceleration is 0.006 m/s^2.
   (d) The raindrop falls vertically from a height of 620 metres. How long will it take the raindrop to fall to ground level?
   (e) A raindrop increases in size as it falls. The volume of a spherical raindrop increases at a rate of 6 cubic millimetres per second. Find the rate at which the radius of the raindrop is increasing when the radius is 1.5mm.

4. **2012 Paper 1**
The functions f and g are defined for $x \in \mathbb{R}$ as
$f(x) : 2x^2 - 3x + 2$ and
$g(x) : x^2 + x + 7$.



(a) Find the co-ordinates of the two points where the curves $y = f(x)$ and $y = g(x)$ intersect.

(b) Find the area of the region enclosed between the two curves.

5. **2014 Sample Paper 1**

(a) Let $f(x) = -0.5x^2 + 5x - 0.98$, where $x \in R$.

i. Find the value of $f(0.2)$.

ii. Show that f has a local maximum point at $(5, 11.52)$.

(b) A sprinter's velocity over the course of a particular 100 metre race is approximately by the following model, where v is the velocity in metres per second, and t is the time in seconds from the starting signal:

$$v(t) = \begin{cases} 0, & : 0 \leq t < 0.2 \\ -0.5t^2 + 5t - 0.98, & : 0.2 \leq t < 5 \\ 11.52, & t \geq 5 \end{cases}$$

Note that the function in part **(a)** is relevant to $v(t)$ above.

i. Sketch the graph of v as a function of t for the first seven seconds of the race.

ii. Find the distance travelled by the sprinter in the first five seconds of the race.

iii. Find the sprinters finishing time for the race. Give your answer correct to two decimal places

(c) A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

i. Prove that the radius of the snowball id decreasing at a constant rate.

ii. If the snowball loses half of its volume in an hour, how long more will it take to melt completely? Give your answer correct to the nearest minute.