Integration
Revision Series 2017

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Introduction

If we have the function $f(x) = x^3$, we say that the derivative of this function is:

$$f'(x) = 3x^2$$

If we start with the function $f'(x) = 3x^2$, we can then say that the **antiderivative** of this function is:

$$f(x) = x^3$$

Integration

Antiderivatives can be calculated by a process called **integration**, which can be seen as a form of reverse differentiation.
The symbol

$$\int f(x)dx$$

called the **indefinite integral**, is used to represent all antiderivatives of $f(x)$.

Integration of Basic Functions

- $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
  (Add one to the power, then put it over the new power)

Some Examples

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$$= \frac{x^4}{2} + x^3 - x^2 + x + c$$

Example

$\int (x^3 + 5x - 4)dx$

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$$\frac{x^4}{4} + \frac{5x^2}{2} - 4x + c$$

Practice

1. $\int (x^2 + 2x + 1)dx$
2. $\int (3x^2 + 4x - 7)dx$
3. $\int (4x^3 - 12x + 17)dx$
4. $\int (3x - x^4)dx$

Answers

1. $\frac{x^3}{3} + x^2 + x + c$
2. $x^3 + 2x^2 - 7x + c$
3. $x^4 - 6x^2 + 17x + c$
4. $\frac{3x^2}{2} - \frac{x^5}{5} + c$

$$\int \frac{1}{x^3} dx$$

As in differentiation, we must rewrite this before we can integrate it.

$$\rightarrow \int x^{-3} dx$$

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This can be rewritten as: $-\frac{1}{2x^2} + c$

Practice

1. $\int \frac{1}{x^4} dx$
2. $\int (\frac{1}{x^2} + \frac{2}{x^3}) dx$
3. $\int \frac{3}{x^2} dx$

Sometimes we have to rewrite, rearrange or factorise a function before we can integrate it. For Example:

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This can now be integrated to:

$$\frac{x^2}{4} + \frac{x}{2} + c$$

Definite Integrals

Given a function $f(x)$ and an interval[a,b], the **definite integral** of $f(x)$ over that interval is given by:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is the antiderivative of f.

Example

$\int_1^2 (3x^2 + 4x)dx$

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We fill in each limit for x and subtract the results.

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Trigonometric Integration

$$\int a\sin(bx)dx \Rightarrow -\frac{a\cos(bx)}{b} + c$$

and

$$\int a\cos(bx)dx \Rightarrow \frac{a\sin(bx)}{b} + c$$

For example:

$$\int 3sin(7x)dx$$

$$\Rightarrow -\frac{3\cos(7x)}{7} + c$$

Products to Sums Formulae

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We use the formula:

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

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$$= \int \frac{1}{2}\left(\sin(5x) + \sin(x)\right)dx$$

$$= -\frac{\cos 5x}{10} - \frac{\cos x}{2} + c$$

Average Value of a Function

To get the average value of a function $f(x)$ on any interval $a \leq x \leq b$ we use the formula:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Average Value

Example: Find the average value of $f(x) = 3x^2 - 5$ on $1 \leq x \leq 3$

Average Value:

$$= \frac{1}{3-1}\int_1^3 (3x^2 - 5)dx$$

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$$= \frac{1}{2} |x^3 - 5x|_1^3$$

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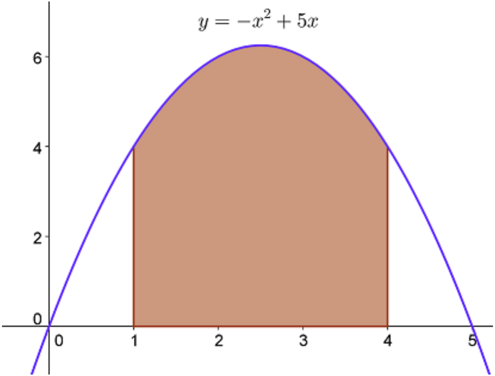
$$= \frac{1}{2}\left(((3)^3 - 5(3)) - ((1)^3 - 5(1))\right)$$

$$= \frac{1}{2}(16)$$

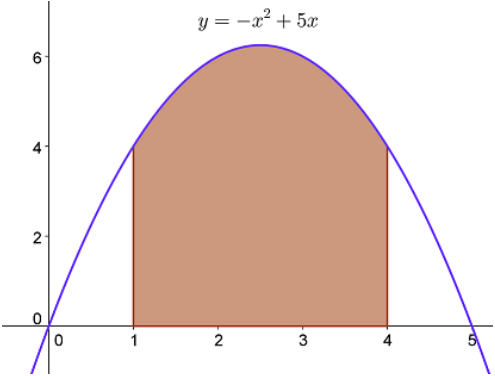
$$= 8$$

Finding Areas by Integration

We can use integration to find the areas bounded by a function and the axes. For example we can find areas such as the shaded region below



Example



To calculate such an area it can be solved by evaluating $\int_1^4 (-x^2 + 5x)dx$

Calculating the Area

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$$= \left| \frac{-x^3}{3} + \frac{5x^2}{2} \right|_1^4$$

$$= \left(\frac{-(4)^3}{3} + \frac{5(4)^2}{2} \right) - \left(\frac{-(1)^3}{3} + \frac{5(1)^2}{2} \right)$$

Calculating the Area

$$\int_1^4 (-x^2 + 5x)dx$$

$$= \left| \frac{-x^3}{3} + \frac{5x^2}{2} \right|_1^4$$

$$= \left(\frac{-(4)^3}{3} + \frac{5(4)^2}{2} \right) - \left(\frac{-(1)^3}{3} + \frac{5(1)^2}{2} \right)$$

$$= \left(\frac{-64}{3} + 40 \right) - \left(\frac{-1}{3} + \frac{5}{2} \right)$$

Calculating the Area

$$\int_1^4 (-x^2 + 5x)dx$$

$$= \left| \frac{-x^3}{3} + \frac{5x^2}{2} \right|_1^4$$

$$= \left(\frac{-(4)^3}{3} + \frac{5(4)^2}{2} \right) - \left(\frac{-(1)^3}{3} + \frac{5(1)^2}{2} \right)$$

$$= \left(\frac{-64}{3} + 40 \right) - \left(\frac{-1}{3} + \frac{5}{2} \right)$$

$$= 16.5$$