8 Sequences and Series

8.1 Arithmetic Series

1. Find T_n, the nth term of each of the following sequences:
   i. 2, 5, 8, 11......
   ii. -8, -4, 0 ,4......
   iii. 9, 5, 1, -3....
2. If T_n of an arithmetic sequence is $3n-2$, find the first term and the common difference.
3. The nth term of a sequence is $3n^2 + 2$. Prove that the sequence is not arithmetic.
4. In an arithmetic sequence $T_3 = 7$ and $T_{10} = 21$. Find T_n, the general term..
5. In an arithmetic sequence, $T_2 + T_3 = 11$ and $T_7 + T_9 = 44$. Find T_n, the general term.
6. Find S_n of the series 3+7+11+15......
   How many terms of the series add to 210?
7. In an arithmetic series, $T_4 = 17$ and $S_6 = 87$. Find the values of a and d.
8. A linear series is such that $S_n = 5n + n^2$.
   (a) Find S_1, S_2 and S_3.
   (b) Hence find the first term and the common difference.
   (c) Find T_n, the general term.
9. Three consecutive terms of a linear series are $5 - x, 2x$ and $3x + 2$. Find the value of x.
10. The sum of the first 4 terms of a linear series is 26. If seven times the third term is equal to four times the fifth term, find T_n.
11. $\log_2 x$, $\log_2 y$ and $\log_2 8$ are three consecutive terms of an arithmetic sequence. If $y = 2x$ find x and y.
12. A concert hall is designed in such a way that the number of seats in a particular row is two more than that of the row in front of it. There are 30 seats in the front row.
    i. How many seats are in the fifth row?
    ii. What row has 50 seats in it?
    iii. If there are 4620 seats in total in the concert hall, how many rows of seats are there?

13. An Irish developer is building a new skyscraper hotel in Dubai. The main hotel lobby and plaza will be an open atrium four floors high. Above the lobby there will be additional floors consisting of regular hotel facilities.
    It will cost a total of €17m to build the first four floors of the building. The fifth floor will cost €0.5m to build, the sixth floor will cost €0.75m to build. Because of the extra height involved in building extra floors, each additional floor will cost €0.25m more that the floor below it.
    i. What will the 45th floor cost to build?
    ii. If the building is to have 90 floors, what would be the total cost of construction?

8.2 Geometric Series

1. Find T_n for each of these geometric sequences:
   i. 3, 6, 12, 24.....
   ii. 72, 36, 18, 9.....
   iii. 18, -6, 2,$-\frac{2}{3}$.....
2. How many terms are in the sequence 64, 32, 16, 8... , $\frac{1}{16}$?
3. Calculate the general term for each of the following geometric series if:
   i. $T_1 = 3$ and $T_4 = 192$
   ii. $T_2 = \frac{9}{2}$ and $T_5 = \frac{1}{6}$
4. Three consecutive terms of a geometric sequence are $x - 7, x - 3$ and $2x$. Calculate the possible values of x.
5. Find S_n of the following geometric series:
   i. 2+6+18.....
   ii. $24 + 6 + \frac{3}{2}$.....
   iii. 1-2+4-8.........
6. Three numbers form a geometric sequence. If their sum is 26 and their product is 216, find the first term and the common ratio.
7. The first three terms of an arithmetic sequence are 12, p and q. The first three terms of a geometric sequence are 16, p and q. Find the value of p and the value of q.
8. The sum of the first five terms of a geometric series is 3124. The fifth term is 25 times the third term. Find the general term (T_n) for the series.
9. The sum of the first three terms of a geometric series is 21. The sum of the first six terms of the same series is 189. Find the first term and the common ratio of the series.
10. A patient takes a drug to relieve epileptic fits. A single dose of the drug contains 50 mg. The concentration of the drug in the bloodstream fades over time. Every hour, the body processes and removes 20% of the drug from the bloodstream.

i. Find a general term to describe the amount of drug in the bloodstream t hours after taking the initial single dose.

ii. What concentration of the drug will be remaining in the bloodstream after 3 hours?

iii. The patient has a very low chance of having an epileptic fit, as long as the concentration of the drug in the bloodstream remains above a critical level of 12 mg. After how many hours should the patient take the second dose?

iv. Against doctors orders, the patient decides to take a dose of the drug every hour. How much of the drug will be in his bloodstream after 4 hours (directly after taking the fifth dose)?

v. A patient will overdose if the concentration of the drug in the blood exceeds 200 mg. If the patient continues to take the drug every hour, how long after the initial dose will he overdose.

11. OPEC control the production of oil in the Middle East, and they have a large amount of control over the price of a barrel of oil. On May 1st the price of a barrel of oil is \$40. OPEC decide to limit production, so the price of a barrel of oil will rise. They predict that the price of oil will rise 2% a day for the month of May.

i. Find a general term that describes the price of a barrel of oil t days after May 1st.

ii. What will the price of a barrel of oil be on May 31st?

iii. An investor realises that OPEC has cut production, so the price of oil will rise. He decides to buy one barrel of oil every day, starting on May 1st, until May 31st (he doesn't buy any oil on May 31st.) What will his total investment in oil be worth on the 31st May?

iv. How much profit would he make if he sold his entire investment in oil on May 31st?

8.3 Infinite Series

1. Calculate the sum to infinity of the following series. (If S_∞ exists)

i. $1 + \frac{1}{2} + \frac{1}{4} + ...$

ii. $1 + 2 + 4 + 8 + ...$

iii. $9 + 3 + 1 + ...$

iv. $8 - 4 + 2 - 1...$

v. $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + ...$

2. Express $0.\dot{6}$ as an infinite geometric series and hence write $0.\dot{6}$ as a rational number.

3. Express $0.\dot{5}\dot{8}$ as a rational number.

4. Express $4.\dot{7}$ as a rational number.

5. $x + \frac{x}{1-x} + \frac{x}{(1-x)^2} + ...$, where $x > 0$, is a geometric series.

i. Find S_∞ of the series in terms of x.

ii. If $S_\infty = 3$, solve for x.

6. $x + \frac{x}{x+2} + \frac{x}{(x+2)^2} + ...$, where $x > 0$, is a geometric series.

 i. Find S_∞ of the series in terms of x.

 ii. If $S_\infty = \frac{15}{4}$, solve for x,

7. For each of the following geometric series, find the range of values of x, for which S_∞ exists.

 i. $1 + \frac{3}{x+1} + \frac{9}{(x+1)^2} + ...$

 ii. $\frac{1}{2} + \frac{1}{3x-1} + \frac{2}{(3x-1)^2} + ...$

 iii. $3 + x + \frac{x^2}{3} + ...$

8. Consider the geometric series: $1 + \cos^2(3\theta) + \cos^4(3\theta) + ...$

 i. If S_∞ of the series is 2, find all possible values of θ, where $0° < \theta < 90°$.

 ii. For what values of θ, $(0° < \theta < 90°)$, would S_∞ be undefined?

9. Consider the geometric series: $\log_4 a + \log_8 a + \log_{\sqrt{512}} a + ...$
 S_∞ of the series is $p \log_4 a$. Find the value of $p \in N$.
 (Hint: Use the change of base rule to express the series in terms of $\log_4 a$)

8.4 Quadratic Sequences and Series

1. Find T_n, the nth term for each of the following quadratic sequences:

 i. 2, 5, 10, 17, 26...

 ii. 6, 12, 20, 30, 42 ...

 iii. 4, 9, 18, 31, 48

2. The number of seats per row in an opera house is described in the table below.

Row	1	2	3	4	5
No. of seats	17	18	21	26	33

 i. Find T_n, which describes the number of seats in row n.

 ii. How many seats are in row 10?

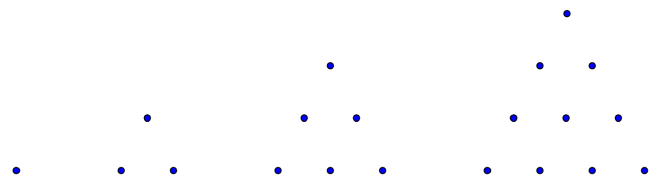
3. The first four terms of a particular pattern are shown in the diagram below (top of next page).

 i. Draw the next pattern.

 ii. Represent the number of dots in each pattern as a series. Find T_n of the series, where T_n is the number of dots in pattern n.

 iii. How many dots are there in the 11th pattern.

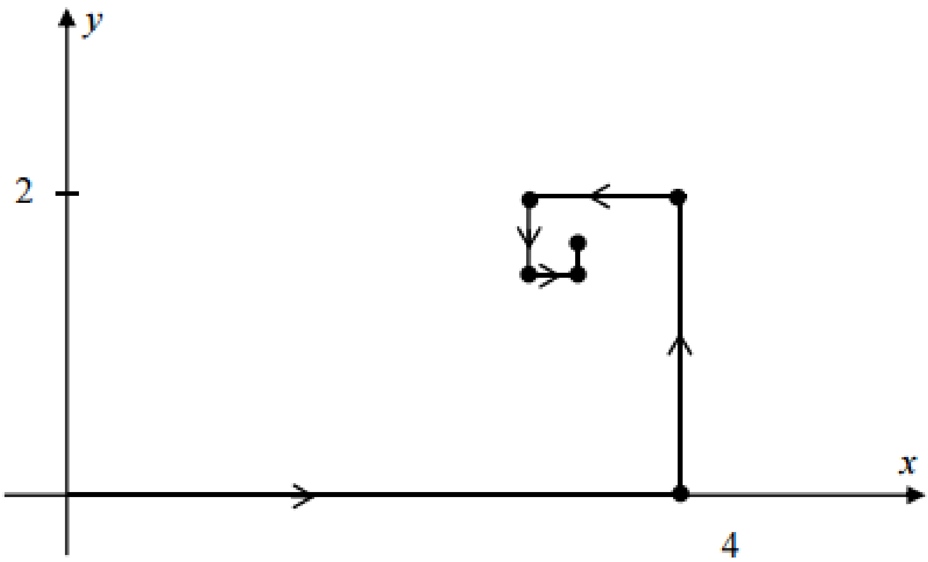
 iv. Which pattern has 210 dots?



8.5 Exam Questions

1. **2016 Paper 1**

   (a) At the first stage of a pattern, a point moves 4 units from the origin in the positive direction along the x-axis. For the second stage, it turns left and moves 2 units parallel to the y-axis. For the third stage, it turns left and moves 1 unit parallel to the x-axis.
   At each stage, after the first one, the point turns left and moves half the distance of the previous stage, as shown



   i. How many stages has the point completed when the total distance it has travelled, along its path, is 7.9375 units?
   ii. Find the maximum distance the point can move, along its path, if it continues in this pattern indefinitely.
   iii. Complete the second row of the table below showing the changes to the x

co-ordinate, the first nine times the point moves to a new position. Hence, or otherwise, find the x co-ordinate and the y co-ordinate of the final position that the point is approaching, if it continues indefinitely in this pattern.

Stage	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**	**8th**	**9th**
Change in *x*	+4	0	–1						
Change in *y*									

(b) A male bee comes from an unfertilised egg, i.e. he has a female parent but he does not have a male parent. A female bee comes from a fertilised egg, i.e. she has a female parent and a male parent.

i. The following diagram shows the ancestors of a certain male bee. We identify his generation as G_1 and our diagram goes back to G_4. Continue the diagram to G_5.

G_1	G_2	G_3	G_4	G_5
			Female	
		Female		
Male	Female		Male	
		Male	Female	

ii. The number of ancestors of this bee in each generation can be calculated by the formula

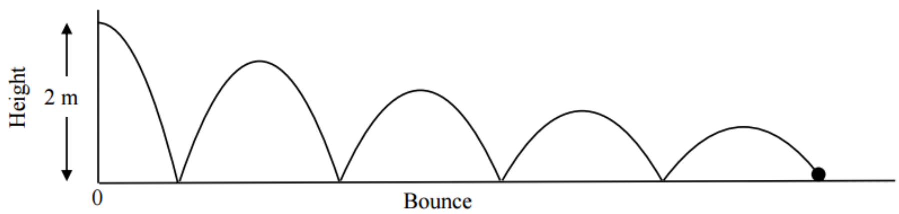
$$G_{n+2} = G_{n+1} + G_n$$

where $G_1 = 1$ and $G_2 = 1$, as in the diagram.
Use this formula to calculate the number of ancestors in G_6 and in G_7.

$$G_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

iii. The number of ancestors in each generation can also be calculated by using the formula Use this formula to verify the number of ancestors in G_3

2. **2015 Paper 1**
Mary threw a ball onto level ground from a height of 2 m. Each time the ball hit the ground it bounced back up to $\frac{3}{4}$ of the height of the previous bounce as shown. 7



(a) Complete the table below to show the maximum height, **in fraction form**, reached by the ball on each of the first four bounces.

Bounce	0	1	2	3	4
Height (m)	$\frac{2}{1}$				

(b) Find, in metres, the total vertical distance (up and down) the ball had travelled when it hit the ground for the fifth time. Give your answer in fraction form.

(c) If the ball were to continue to bounce indefinitely, find, in metres, the total vertical distance it would travel.

3. **2014 Paper 1** The n^{th} term of a sequence is $T_n = lna^n$, where $a > 0$ and a is a constant.

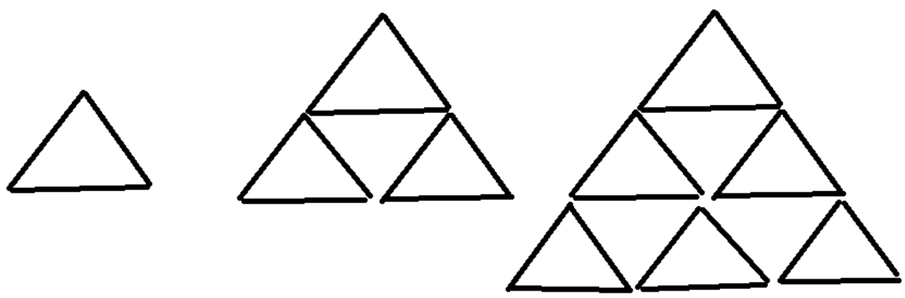
   (a) i. Show that T_1, T_2 and T_3 are in arithmetic sequence.

   ii. Prove that the sequence is arithmetic and find the common difference.

   (b) Find the value of a for which $T_1 + T_2 + T_3 + ... + T_{98} + T_{99} + T_{100} = 10100$

   (c) Verify that for all values of a,
   $(T_1 + T_2 + T_3 + ... + T_{10}) + 100d = (T_{11} + T_{12} + T_{13} + ... + T_{20})$
   Where d is the common difference of the sequence.

4. **2013 Paper 1** Shapes in the form of small equilateral triangles can be made using matchsticks of equal length. These shapes can be put together into patterns. The beginning of a sequence of these patterns is shown below.



   (a) i. Draw the fourth pattern in the sequence.

   ii. The table below shows the number of small triangles in each pattern and the number of matchsticks needed to create each pattern. Complete the table.

   (b) Write an expression in n for the number of triangles in the n^{th} pattern in the sequence.

Pattern	1^{st}	2^{nd}	3^{rd}	4^{th}
Number of small triangles	1		9	
Number of matchsticks	3	9		

(c) Find an expression, in n, for the number of matchsticks needed to turn the $(n-1)^{th}$ pattern into the n^{th} pattern.

(d) The number of matchsticks in the n^{th} pattern in the sequence can be represented by the function $u_n = an^2 + bn$ where $a, b \in \mathbb{Q}$ and $n \in \mathbb{N}$. Find the value of a and the value of b.

(e) One of the patterns in the sequence has 4134 matchsticks. How many small triangles are in that pattern?

5. **2012 Paper 1**

   (a) Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:
   $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1-r^n)}{1-r}$.

   (b) By writing the recurring part as an infinite geometric series, express the following number as a fraction of integers: $5.\dot{2}\dot{1} = 5.2121212121...$

6. **2012 paper 1** The *atmospheric pressure* is the pressure exerted by the air in the earth's atmosphere. It can be measured in kilopascals (kPa). The average atmospheric pressure varies with altitude: the higher up you go, the lower the pressure is. Some students are investigating this variation in pressure, using some data that they found on the internet. They have information about the average pressure at various altitudes.
   Six of the entries in the data set are as shown in the table below: By looking at the

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101.3	89.9	79.5	70.1	61.6	54.0

   pattern, the students are trying to find a suitable model to match the data.

   (a) Hannah suggests that this is approximately a geometric sequence. She says she can match data fairly well by taking the first term as 101.3 an the common ratio as 0.883.

      i. Complete the table and show the values given by Hannah's model, correct to one decimal place.

altitude (km)	0	1	2	3	4	5
pressure (kPa)	101.3					

      ii. By considering the percentage errors in the above values, insert an appropriate number to complete the following statement: "Hannah's model is accurate within.......%"

(b) Thomas suggest modelling the data with the following exponential function: $p = 101.3 \times^{-0.1244h}$
Where p is the pressure in kilopascals, and h is the altitude in kilometres.
   i. Taking any one of the values other than zero for the altitude, verify that the pressure given by Thomas's model and the pressure given by Hannah's model differ by less than 0.01 kPa.
   ii. Explain how Thomas might have arrived at the value of the constant 0.1244 in his model.

(c) Hannah's model is *discrete*, while Thomas's is *continuous*.
   i. Explain what this means.
   ii. State one advantage of a continuous model over a discrete one.

(d) Use Thomas's model to estimate the atmospheric pressure at the altitude of the top of Mount Everest: 8848 metres.

(e) Using Thomas's model, find the estimate for the altitude at which the atmospheric pressure is half of its value at sea level (altitude 0km).

(f) People sometimes experience a sensation in their ears when pressure changes. This can happen when travelling in a fast lift in a tall building. Experiments indicate that many people feel such a sensation if the pressure changes rapidly by 1 kilopascal or more. Suppose that such a person steps into a lift that is close to sea level. Taking a suitable approximation for the distance between two floors, estimate the number of floors that the person would need to travel in order to feel this sensation.