Leaving Certificate Examination, 2015

Sample paper prepared by Leamy Maths Community

Mathematics

Project Maths - Phase 3

Paper 1

Solutions

Higher Level

Sunday 19 April

Paper written by J.P.F. Charpin

Leamy Maths Community

300 marks

Section A Concepts and Skills 150 Marks

Answer **all six** questions from this section.

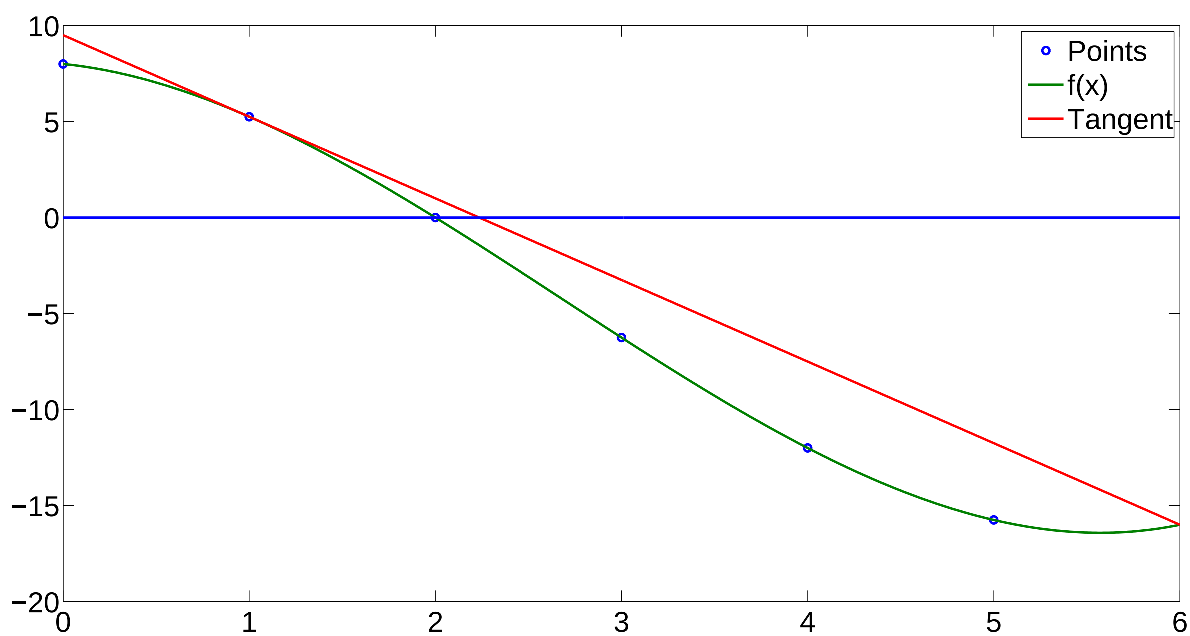
Question 1 **(25 Marks)**

(a) Plot the curve

$$f(x) = \frac{1}{4}x^3 - 2x^2 - x + 8$$

x	0	1	2	3	4	5
f(x)	8	$\frac{21}{4}$	0	-$\frac{25}{4}$	-12	-$\frac{63}{4}$

5 marks



(b) Explain why (x-2) is a factor of the function f and find the other two roots of the function f(2)=0 so (x-2) is a factor of the function f. Using long division, you get

$$\begin{aligned} f(x) &= \frac{1}{4}(x-2)\left(x^2-6x-16\right) \\ &= \frac{1}{4}(x-2)(x+2)(x-8) \end{aligned}$$

x=-2 and x=8 are the two other roots.
2+5+3 marks

(c) Calculate the equation of the tangent to the curve at $x=1$ and plot it on the same graph as the function

$$\begin{aligned} y &= f(1)+f'(1)(x-1) \\ f'(x) &= \frac{3}{4}x^2-4x-1 \qquad \Longrightarrow \qquad f'(1)=-\frac{17}{4} \\ y &= \frac{21}{4}-\frac{17}{4}(x-1)=-\frac{17}{4}x+\frac{19}{2} \end{aligned}$$

5+5 marks

Question 2 **(25 Marks)**

(a) Write the complex number

$$z_1=\frac{-1}{1+i}$$

in the form $a+ib$ where a and b are real numbers and calculate its polar form.

$$\begin{aligned} z_1 &= -\frac{1}{2}+\frac{1}{2}i \\ &= \frac{\sqrt{2}}{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right) \end{aligned}$$

4+4 marks

(b) Calculate $\bar{z_1}$, $z_2=2iz_1$ and $z_3=-\frac{i}{2}z_1$.

$$\begin{aligned} \bar{z_1} &= -\frac{1}{2}-\frac{i}{2} \\ z_2 &= 2iz_1=-1-i \end{aligned}$$

$$z_3=-\frac{i}{2}z_1=\frac{1}{4}+\frac{i}{4}$$

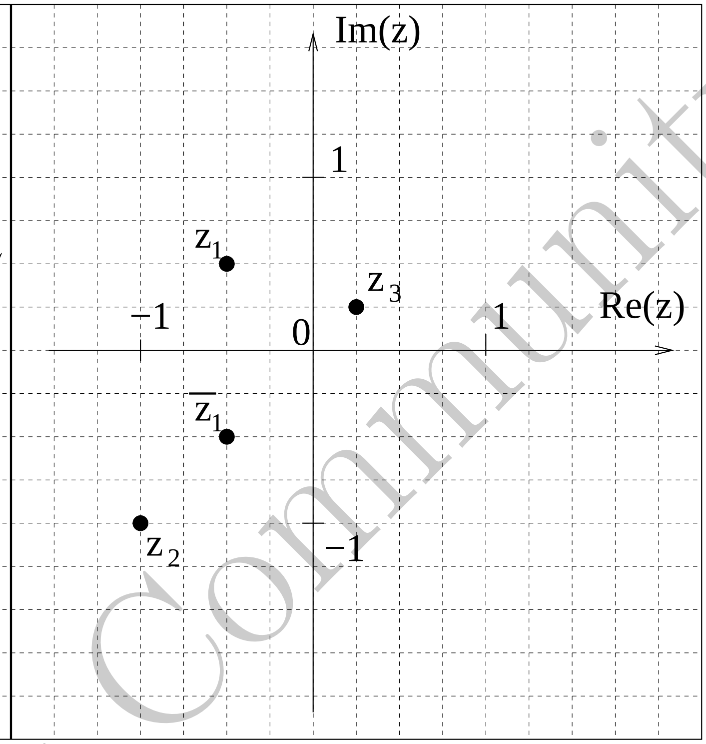
5 marks

(c) Plot z_1, $\bar{z_1}$, z_2 and z_3 on the Argand diagram. What are the three geometrical transformation observed when transforming z_1 in $\bar{z_1}$, z_1 in z_2 and z_1 in z_3? 4+3 marks

z_1 to $\bar{z}_1$: Reflexion about the x axis

z_1 to z_2 : Dilation of 2 and rotation $\pi/2$

z_1 to z_3: Dilation of 1/2 and rotation $-\pi/2$



(d) Calculate z_1^5 and $\bar{z_1}^5$ in Cartesian form.

$$z_1^5 = \frac{1}{8} - \frac{i}{8} \quad \bar{z_1}^5 = \frac{1}{8} + \frac{i}{8}$$

5 marks

Question 3 **(25 Marks)**

(a) Show that

$$n(n+1)(2n+1) + 6(n+1)^2 = (n+1)(n+2)(2n+3)$$

$$\begin{aligned}
& (n+1)(n+2)(2n+3) = 2n^3 + 9n^2 + 13n + 6 \\
& n(n+1)(2n+1) + 6(n+1)^2 = 2n^3 + 9n^2 + 13n + 6 \\
\text{Conclusion} \quad & n(n+1)(2n+1) + 6(n+1)^2 = (n+1)(n+2)(2n+3)
\end{aligned}$$

5 marks

(b) Using this result, show by induction that for $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

P(n) is the property

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- P(1) is true as $1^2 = 1$ and 1(1+1)(2(1)+1)/6=1
- Assume P(n) is true

$$
\begin{aligned}
& 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} \\
\Longrightarrow\ & 1^2+2^2+3^2+\ldots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6}+(n+1)^2 \\
\Longrightarrow\ & 1^2+2^2+3^2+\ldots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \\
\Longrightarrow\ & 1^2+2^2+3^2+\ldots+n^2+(n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}
$$

  This is the property at level n+1
- The property is inductive, true at level n=1 so it is true for all $n \geq 1$

15 marks

Explain why

$$2^2+4^2+6^2+\ldots+(2n)^2 = 4\left(1^2+2^2+3^2+\ldots+n^2\right)$$

and then show that

$$2^2+4^2+6^2+\ldots+(2n)^2 = \frac{2}{3}n(n+1)(2n+1)$$

and calculate $2^2+4^2+6^2+\ldots+100^2$. Detail your calculations.

$$
\begin{aligned}
2^2+4^2+6^2+\ldots+(2n)^2 &= 4(1)^1+4(2)^2+4(3)^2+\ldots+4(n)^2 \\
&= 4\left(1^2+2^2+3^2+\ldots+n^2\right) \\
&= 4\times\frac{1}{6}n(n+1)(2n+1) \\
&= \frac{2}{3}n(n+1)(2n+1)
\end{aligned}
$$

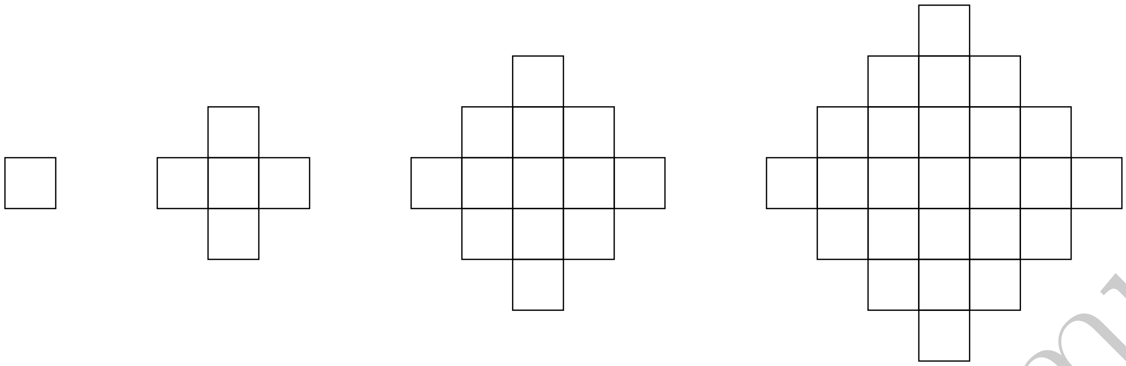
The expression corresponds to $n = 50$ so

$$2^2+4^2+6^2+\ldots+100^2 = \frac{2}{3}(50)(50+1)+(2(50)+1) = 171,700$$

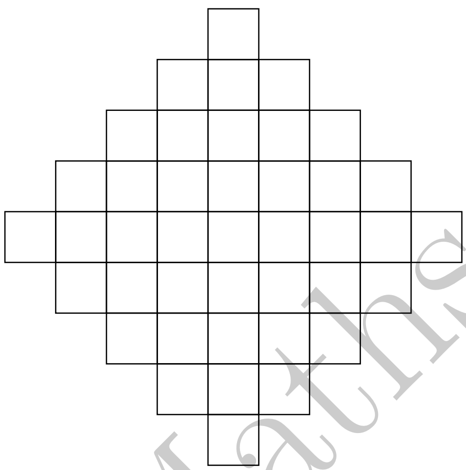
1+2+2 marks

Question 4 **(25 Marks)**

(a) Tiles are arranged in the pattern shown below. Draw the next pattern and complete the table



n=1 n=2 n=3 n=4



n=5

n	1	2	3	4	5
T_n	1	5	13	25	41

2+3 marks

(b) Prove that the series may be described by

$$T_n = a\ n^2 + b\ n + c$$

and calculate the values of a, b and c.
The second difference is constant equal to 4 so $2a = 4 \Longrightarrow a = 2$, leading to $T_n = 2n^2 + bn + c$.

$$\begin{aligned} & T_1 = 1 = 2 + b + c \qquad T_2 = 8 + 2b + c \\ \Longrightarrow\ & b + c = -1 \qquad 2b + c = -3 \\ \Longrightarrow\ & b = -2 \qquad c = 1 \end{aligned}$$

$$T_n = 2\ n^2 - 2\ n + 1$$

10 marks

(c) How many tiles are necessary at level 10?

$$T_{10} = 2(10)^2 + 2(10) + 1 = 181$$

4 marks

(d) 1000 tiles are available. What will be the last level of the sequence completed? How many tiles will be left? The level n corresponding to 1000 tiles verifies

$$2n^2 - 2n - 999 = 0 \Longrightarrow n = 22.85 \quad or \quad n = -21.85$$

n=-21.85 is impossible, all values between 0 and 22.85 are possible. The last completed level is 22, $T_{22} = 925$ and there are 75 tiles left.
6 marks

Question 5 **(25 Marks)**

(a) Solve the following inequalities

(i)

$$x^2 - 4x > -3$$

$x < 1$ and $x > 3$
5 marks

(ii)

$$\frac{x+2}{x+5} > -1$$

$x < -5$ and $x > -7/2$
5 marks

(iii)

$$\frac{1}{4} \geq \left(\frac{4}{5}\right)^x$$

$$x \geq \frac{\ln(1/4)}{\ln(4/5)} = -\frac{\ln 4}{\ln 4 - \ln 5} = 6.21$$

5 marks

(b) Solve the simultaneous equations

$$\begin{cases} 2x + 3y + 4z = -23 \\ x + 3y - 5z = 25 \\ -3x + 2y + 4z = -9 \end{cases}$$

x=-3, y=1, z=-5
10 marks

Question 6 **(25 Marks)**

(a) Consider the function

$$f(x) = 4cos(3x)$$

(i) What is the domain and range of this function?
Domain= $\mathbb{R}$, Range=[-4:4]

(ii) Is the function periodic? If so give the period. Justify your answer.
$\cos x$ is 2π periodic so $\cos(3x$ is $2\pi/3$ periodic.
5 marks

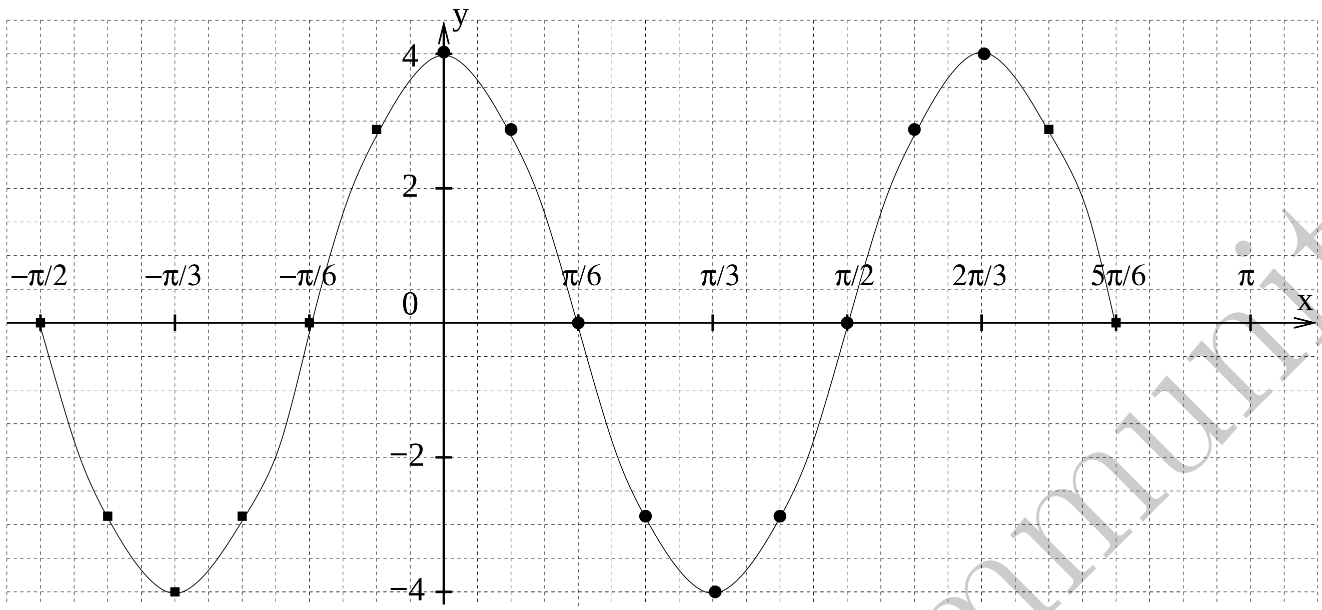
(iii) Is the function injective, surjective or bijective? If not provide the largest possible domain on which one of these properties is true. Justify your answers.

- Not injective as $f(x) = 0$ has more than one solution
- Not surjective as $f(x) = 5$ has no solution
- for example, bijective from $[0, \pi/3]$ to [-4:4],
- Surjective from $\mathbb{R}$ to [-4:4]

5 marks

(b) Complete the table below and using the information above, draw the curve for $f(x)$. Explain why the information in the table is sufficient to draw the curve.

x	0	$\pi/12$	$\pi/6$	$3\pi/12$	$\pi/3$	$5\pi/12$	$\pi/2$	$7\pi/12$	$2\pi/3$
$f(x)$	4	$2\sqrt{2}$	0	-2$\sqrt{2}$	-4	-2-$\sqrt{2}$	0	$2\sqrt{2}$	4



The square points may be found using the periodicity of the function
5 marks

(c) Calculate the value of the function f on the interval $[0,\ \pi/6]$.

$$\bar{f} = \frac{6}{\pi}\int_0^{\frac{\pi}{6}} 4\cos(3x)dx = \frac{8}{\pi}$$

10 marks

Section B Contexts and Applications 150 Marks

Answer **all three** questions from this section.

Question 7 **(50 Marks)**

Aoife takes out a ten year loan at the bank to improve her home. She needs €50,000.

(a) Every month, Aoife pays an interest rate of 0.25%

(i) Calculate the annual equivalent interest rate

$$AER = \left(1 + \frac{0.25}{100}\right)^{1} 2 - 1 = 3.042\%$$

(ii) Another bank offers Aoife a 10 year rate of 35%. What should she do?

$$1.35 = (1 = i)^{10} \implies i = 1.35^{1/10} - 1 = 3.047\%$$

She should take the first one
5 marks

(iii) Aoife takes her loan at the annual rate of r=3%. Calculate how much she will pay every year

$$A = 50,000\frac{0.03(1.03)^{1}0}{(1+0.03)^{10} - 1} = e5,862.52$$

5 marks

(iv) Assuming inflation is 2%, calculate the present value at the beginning of the ninth year of the payment to be made at the end of year 9 and the payment at the end of year 10. To the nearest €, how much should Aoife be charged to repay her loan at the beginning of year 9?

$$PV_1 = \frac{5862}{1.02} \approx 5,747$$
$$PV_2 = \frac{5862}{1.02^2} \approx 5,634$$

She should pay 5747+5634=€11,381
5 marks

(b) If you payback a loan early, banks charge penalties. If Aoife wants to clear her loan at the beginning of year 9, she will have to pay back one lump payment of €13,000.
At the beginning of every year she can save €2,500 and she puts this money on an account earning 2.5% a year.

(i) Calculate the amount on the account at the end of the first and second year year.

$$\begin{aligned} Y_1 &= 2,55(1.025) = 2562.5 \\ Y_2 &= (2562 + 2500)(1.025) = 5189.06 \end{aligned}$$

3 marks

(ii) Show how to use the sum of a geometric series to calculate the value of Aoife's investment after n years.
At the end of Year 1, you have

$$Y_1 = 2,500(1.025)$$

At the end of Year 2, you have

$$Y_2 = 2,500(1.025)^2 + 2,500(1.025)$$

at the end of year n, you have

$$Y_n = 2,500(1.025)^n + 2,500(1.025)^{n-1} + 2,500(1.025)$$

This is a geometric series and it adds up to

$$Y_n = 2,500(1.025)\frac{1.025^n - 1}{1.025 - 1}$$

10 marks

(iii) Neglecting interest rates, estimate after how many years Aoife should start saving to have the €13,000 at the beginning of year 9.
She should pay during $13,000/2,500 \approx 5$ years and a few months: she has to save for 6 years
2 marks

(iv) Using trial and error, calculate the minimum number of years she should be saving to reach €13,000 now including interest rates.

$$Y_5 = 13,469.34$$

She only needs to save for 5 years
5 marks

(v) Verify your answer by using the formula you found in part b(ii)?

$$13,000 = 2,500(1.025)\frac{1.025^n - 1}{1.025 - 1} \Longrightarrow n = \left[\ln\left(\frac{13000}{2500(1.025)}(1.025-1)+\right)\right] / \ln(1.025) \approx 4.83$$

She needs to save for 5 years
5 marks

(vi) Why are the two numbers different? Interest rates are not taken into account in (iii)
2 marks

(c) Aoife will actually deposit a fixed amount every month on her savings account at 2.5% instead of a lump sum at the beginning of every month.

(i) Find the monthly interest rate with the rate in % correct to 3 decimal places.

$$1.025 = (1+i)^12 \Longrightarrow i = 1.025^{1/12} - 1 = 0.0206\%$$

3 marks

(ii) Correct to 2 decimal places, how much should Aoife pay every month reach €2500 at the end of the first year?
using the same formula as in b(ii)

$$P = A(1+i)\frac{(1+i)^{12} - 1}{i} \Longrightarrow A = \frac{Pi}{(1+i)\left[(1+i)^12 - 1\right]} = e205.56$$

5 marks

Question 8 **(50 Marks)**

A chef prepares a dish which needs to be served at a temperature higher than $30°C$. The temperature of this food is modelled using the formula

$$T = T_a + ae^{bt}$$

where

- T_a represents the ambient temperature,
- T is the temperature of the dish,
- t is the time in minutes.

The ambient temperature in the kitchen is $20°C$. When the chef finishes the dish, its temperature is $50°C$. After 1 minute, the temperature has fallen to $45°C$

(a) When the chef finishes the dish, its temperature is $50°C$. After 1 minute, the temperature has fallen to $45°C$

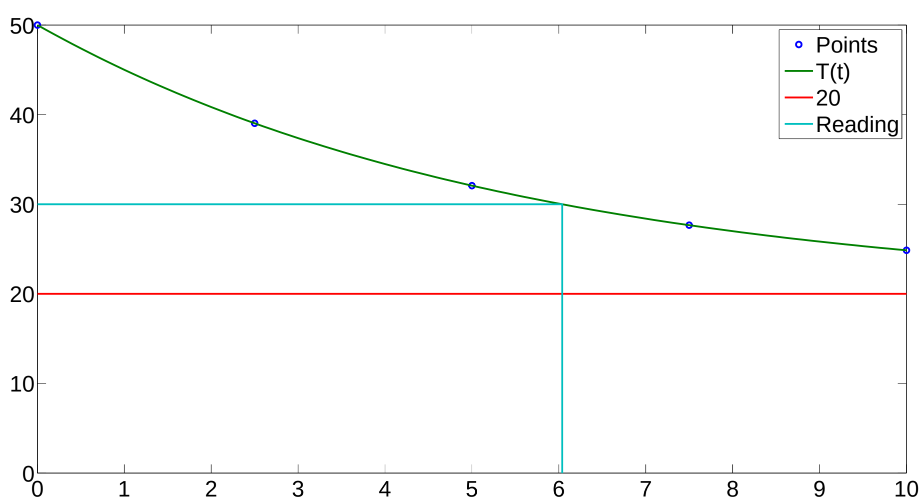
(i) What is the value of T_a? What can you say about the sign of b (justify your answer)?
$T_a = 20$
As the temperature of the dish is larger that the temperature of the room, a¿0 so b¡0. as the temperature of the dish will decrease.
5 marks

(ii) Using the information provided, calculate the values for a and b and give the expression for the temperature of the dish

$$T = 20 + 30e^{-0.182t}$$

10 marks

(iii) Plot the temperature for $t \in [0 : 10]$ using points corresponding to t=0, 2.5, 5, 7.5 and 10 minutes



10 marks

(iv) Using the graphic, estimate at what time the temperature of the dish will reach $30°C$. Explain how you read the result on the graph.
Read approximately 6 minutes
5 marks

(v) Calculate the exact result using the formula you found for the temperature of the dish.

$$t = -\frac{\ln 3}{-0.182} = 6.04\ min$$

5 marks

(b) The chef uses a heating device to make sure that the temperature of the dish does not fall below $30°C$. The temperature of the dish is now provided by

$$T = 20 + 30e^{-0.182t} + ct$$

where c represent the intensity of the heating device.

(i) Show the the minimum temperature is reached at time

$$t_m = \frac{1}{0.182} \ln\left(\frac{5.46}{c}\right)$$

and justify that this is a minimum.

$$T'(t) = 30(-0.182)e^{-0.182t} + c$$
$$T'(t) = 0 \Longrightarrow t = \frac{1}{0.182} \ln\left(\frac{5.46}{c}\right)$$
$$T''(t) = 30(-0.182)^2 > 0$$

Since T'' is positive, this is a minimum
10 marks

(ii) The chef can choose between two settings, corresponding to $c = 0.5$ and $c = 1$. Using the result of the previous question to justify your answer, what value should he choose and what would be the corresponding minimum temperature?

- c=0.5, T_{min}=29.31, $t_{min} = 13.13$
- c=1, T_{min}=34.82, $t_{min} = 9.32$

Choose $c = 1$, the temperature will always be above 30
5 marks

Question 9 **(50 Marks)**

The number of vehicles per hour between 05.00 am and 14.00 pm at a given spot on busy stretch of road is given by the function

$$f(t) = \begin{cases} 20(t-5)^2(3t^2 - 62t + 325) & if \quad 5 \le t \le 10 \\ 2500 & if \quad 10 \le t \le 14 \end{cases}$$

Times are in hours. $t = 6$ correspond to 06.00am, $t = 8.5$ corresponds to 8.30am. In the following, you may write all times in the decimal form, using 8.5 for 08.30am.

(a) Morning rush

(i) Show that the derivative of the function for $5 \le t \le 10$ may be written as

$$f'(t) = (t-5)(at^2 + bt + c)$$

$$\begin{aligned} f'(t) &= 20(2t-10)\left(3t^2 - 62t + 325\right) + 20(t-5)^2(6t-62) \\ &= 20(t-5)\left(6t^2 - 124t + 650 + (t-5)(6t-62)\right) \\ &= 20(t-5)\left(6t^2 - 124t + 650 + 6t^2 - 92t + 310\right) \\ &= 20(t-5)\left(12t^2 - 216t + 960\right) \\ &= 240(t-5)\left(t^2 - 18t + 80\right) \end{aligned}$$

7 marks

(ii) Identify the times at which the traffic reaches minimum and maximum values.

$$\begin{aligned} f'(t) = 0 \quad &\implies \quad t - 5 = 0 \qquad t^2 - 18t + 80 = 0 \\ &\implies \quad t = 5, \quad t = 8, \quad t = 10 \\ f''(t) \quad &= \quad 240\left(t^2 - 18t + 80\right) + 240(t-5)(2t-18) \\ &= \quad 240\left(3t^2 - 46t + 170\right) \end{aligned}$$

$f''(8) < 0$ so t=8 is a maximum.
$f''(5) > 0$ and $f''(10) > 0$ so t=5 and t=10 are a minimum.
8 marks

(iii) Calculate the corresponding value of the number of vehicles per hour.

$$f(5) = 0 \quad f(8) = 3780 \quad f(10) = 2500$$

5 marks

(b) The number of vehicles using the road between the times $t = a$ and $t = b$ is given by the integral

$$N = \int_a^b f(t)dt$$

(i) Show that

$$20(t-5)^2(3t^2-62t+325) = 60t^4 - 1840t^3 + 20400t^2 - 96000t + 162500$$

$$\begin{aligned} 20(t-5)^2(3t^2-62t+325) &= 20(t^2-10t+25)(3t^2-62t+325) \\ &= 20(3t^4-92t^3+1020t^2-4800t+8125) \\ &= 60t^4-1840t^3+20400t^2-96000t+162500 \end{aligned}$$

5 marks

(ii) Show that 12,500 vehicles use the road between 5.00am and 10.00am.

$$\begin{aligned} N &= \int_5^{10} \left(60t^4-1840t^3+20400t^2-96000t+162500\right) dt \\ &= 60\left[\frac{t^5}{5}\right]_5^{10} - 1840\left[\frac{t^4}{4}\right]_5^{10} + 20400\left[\frac{t^3}{3}\right]_5^{10} - 96000\left[\frac{t^2}{2}\right]_5^{10} + 162500\,[t]_5^{10} \\ &= 1162500 - 4312500 + 5950000 - 3600000 + 812500 \\ &= 12500 \end{aligned}$$

15 marks

(iii) Due to road works, only 18,000 vehicles will be able to use the road between 5.00am and 14.00pm. At what time will this number be reached?
18000-12500=5500. There are 2500 vehicles per hour so 5500 will be reached after t=5500/2500=2.2 hours, this is 2 hours and 12 minutes. This happens at 12.12.
5 marks

(c) Vehicles will now be charged when using this portion of the road. Drivers will need to pay €8 before 10.00am and €4 between 10.00am and 14.00pm. Calculate the revenue generated if there are no road works.

- Before 10.00 you earn: 12500(8)=€100000
- Between 10.00 and 14.00 you earn: 2500(4)(4)=€40000

In total, you earn €140000.
5 marks