Leaving Certificate Examination, 2016

Sample paper prepared by Leamy Maths Community

Mathematics

Paper 1

Higher Level

Sunday 24 April

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Leamy Maths Community

Solutions

300 marks

Sample Instructions

There are two sections in this examination paper:

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer questions as follows:

In Section A, answer all six questions.
In Section B, answer all three questions.

Write your answers in the spaces provided in this booklet. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the booklet of *Formulae and Tables*. You must return it at the end of the examination.

You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A Concepts and Skills 150 Marks

Answer **all six** questions from this section.

Question 1 (25 Marks)

The number of seats in an opera house is described in the table below

Row	1	2	3	4	5
No. of seats	17	21	25	29	33

(a) Initial design

(i) Calculate the common difference and show that the number of seats is an arithmetic series.

$$T_n = a + b \times n$$

Calculate the values of a and b. You must use this formula in the following to get full marks.
Common difference =4 hence form is as given.

$$T_1 = a + b = 17 \qquad T_2 = a + 2b = 21 \Longrightarrow a = 13, \qquad b = 4$$

Marks: 0, 2, 5

(ii) How many seats are there in the 15-th row?

$$T_{15} = 13 + 15(4) = 73$$

(iii) Calculate the total number of seats if there are 15 rows.

$$S_{15} = (13)(15) + 4\frac{(15)(16)}{2} = 675$$

Marks: 0, 2, 3, 5

(iv) How many rows do you need if there are to be over 800 seats in the opera house.

$$800 = 13n + 2n(n+1) \qquad 2n^2 + 15n - 800 = 0 \Longrightarrow n = \frac{-15 + \sqrt{15^2 + 8(800)}}{4} = 16.6$$

17 rows
Marks: 0, 4, 7, 10

(b) The architect decides to change the design. The number of seats per row is now:

Row	1	2	3	4	5
No of seats	17	18	21	26	33

(i) Calculate the second common difference and without calculating the values of a, b and c, show that the number of rows may be written as:

$$T_n = a + b \times n + c \times n^2$$

SCD=2 hence the format

(ii) Calculate the number of seats in row 10 if $a = 18$, $b = -2$ and $c = 1$.

$$T_3 = 18 - 2(1) + 10^2 = 98$$

Marks: 0, 2, 5

Question 2 (25 Marks)

Consider the function f

$$f(x) = x^3 - 4x^2 + ax + 10$$

(a) Special case $a = -7$

(i) Plot the function for $a = -7$ and $-2.5 \leq x \leq 5.5$.

-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5
-13.125	0	8.125	12	12.375	10	5.625	0	-6.125

2	2.5	3	3.5	4	4.5	5	5.5	
-12	-16.875	-20	-20.625	-18	-11.375	0	16.875	

Marks: 0, 4, 7, 10

(ii) Using your graph, find the three roots of equation $f(x) = 0$. Check your results using the function definition.
$x = -2$, $x = 1$, $x = 5$

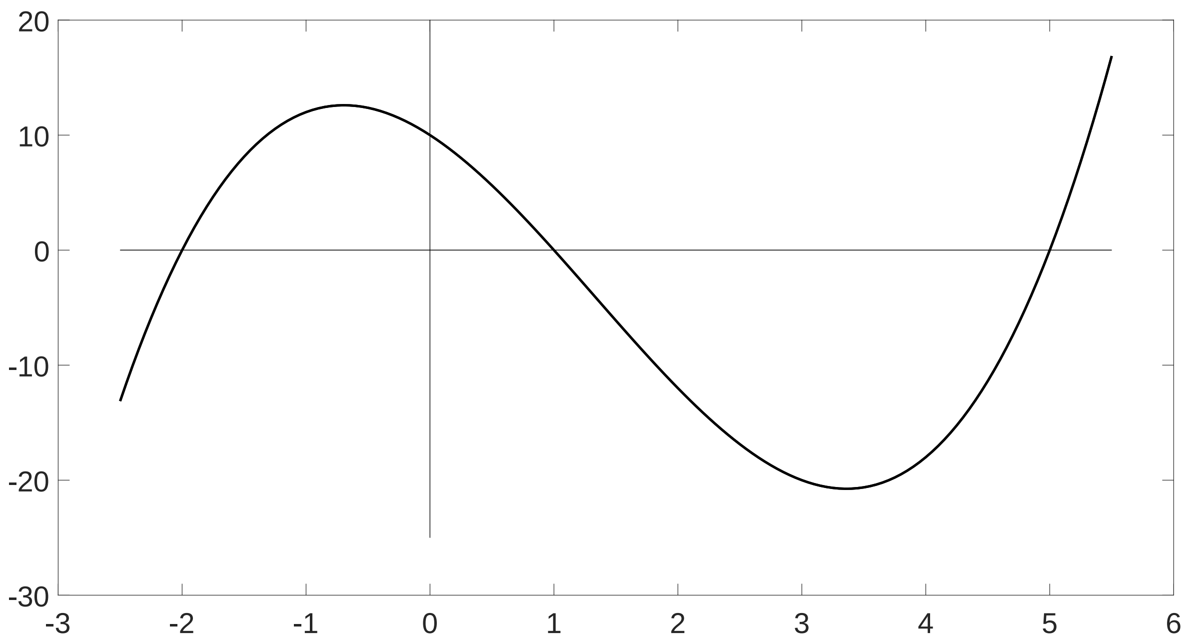
(iii) Hence write $f(x)$ in its factorised form.
$f(x) = (x + 2)(x - 1)(x - 5)$
Marks: 0, 2, 5

(b) Increasing function

(i) Calculate the derivative of function f for a general value of the parameter a.

$$f'(x) = 3x^2 - 8x + a$$

Marks: 0, 2, 5



(ii) For what values of a in the function always increasing?

$$f'(x) > 0 \Longrightarrow 64 - 12a < 0 \Longrightarrow a > \frac{16}{3}$$

(iii) Is this result compatible with what you found in part (a)?
Yes as $-7 < 16/3$
Marks: 0, 2, 5

Question 3 **(25 Marks)**

Consider the equation

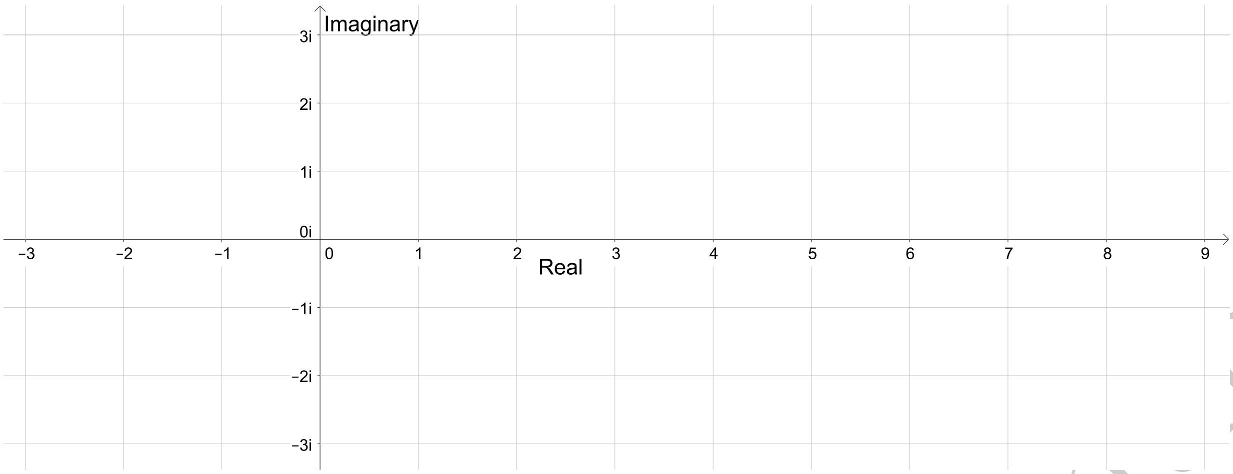
$$z^3 - 7z^2 + 17z - 15 = 0$$

(a) If $z_1 = 2 - i$ is a solution, find the other two solutions of the equation
$z = 2 - i$, $z = 2 + i$, $z = 3$
Marks: 0, 4, 7, 10

(b) Place the three roots on the Argand diagram

Marks: 0, 2, 5

(c) What geometrical operation can be used to transform z_1 into the other root with a non zero imaginary part?
Symmetry with respect to the x axis
Marks: 0, 2, 5



(d) Write the number $z_1 = 2 - i$ in polar form. Express the argument in degrees, correct to 3 decimal places.

$$|z_1| = \sqrt{2^2 + (-1)^2} = \sqrt{5} \qquad \theta = \tan^{-1}\left(-\frac{1}{2}\right) = -26.565$$

Marks: 0, 2, 5

Question 4 **(25 Marks)**

(a) Solve the following inequalities.

(i) Find x if

$$\frac{x-2}{x-1} < 2 \implies (x-1)(x-2) < 2(x-1)^2$$
$$\implies 0 < x^2 - x$$
$$\implies x < 0 \qquad x > 1$$

Marks: 0, 3, 5, 7

(ii) Find x if

$$\left(\frac{1}{3}\right)^x < 9 \implies x \ln\left(\frac{1}{3}\right) < 2\ln 3$$
$$\implies = x \ln 3 < 2 \ln 3$$

$x > -2$

Marks: 0, 2, 4, 6, 8

(b) Solve the system:

$$\begin{aligned} 2x + y - z &= 7 \\ x + 3y - 2z &= 13 \\ 4x - 5y - 3z &= 3 \end{aligned}$$

$x = 1,\ y = 2,\ z = -3$

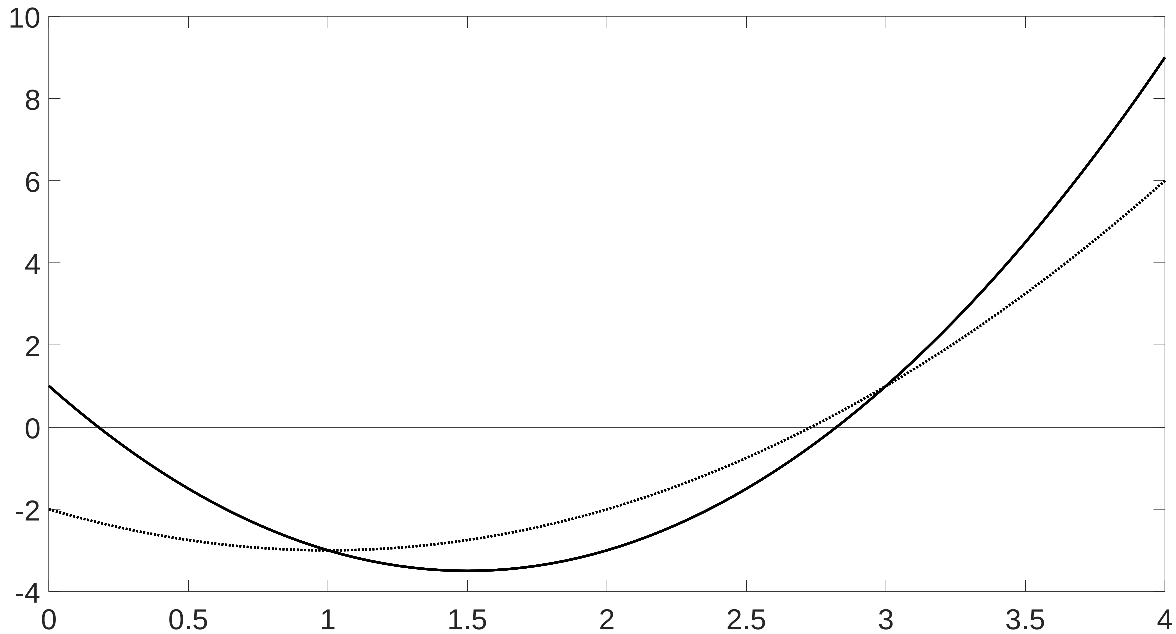
Marks: 0, 4, 7, 10

Question 5 (25 Marks)

(a) Plot the two functions f and g on the graph below.

$$f(x) = 2x^2 - 6x + 1 \qquad g(x) = x^2 - 2x - 2$$

x	0	0.5	1	1.5	2	2.5	3	3.5	4
f	1	-1.5	-3	-3.5	-3	-1.5	1	4.5	9
g	-2	-2.75	-3	-2.75	-2	-0.75	1	3.5	6



Marks: 0, 4, 7, 10

(b) Read on the graph where the two functions cross and verify this result analytically.

$$2x^2 - 6x + 1 = x^2 - 2x - 2 \Longrightarrow x^2 - 4x + 3 = 0 \longrightarrow x = 1 \qquad x = 3$$

Marks: 0, 2, 5

(c) Shade the area between the two curves and use calculus to calculate the value of the area.

$$\begin{aligned} \int_1^3 (g-f)dx &= \left[-\frac{x^3}{3}\right]_1^3 + 2\left[x^2\right]_1^3 - 3\left[x\right]_1^3 \\ &= \frac{-27+1}{3} + 2(9-1) - 3(3-1) = \frac{4}{3} \end{aligned}$$

Marks: 0, 4, 7, 10

Question 6 (25 Marks)

(a) Show that the second order derivative of function

$$f(x) = e^{-x^2/2}$$

is

$$f''(x) = (x^2 - 1)e^{-x^2/2} .$$

$$f'(x) = -xe^{-x^2/2} \qquad f''(x) = (x^2 - 1)e^{-x^2/2} .$$

Marks: 0, 4, 7, 10

(b) Hence calculate the turning point(s) and inflexion point(s).

$$f'(x) = 0 \Longrightarrow x = 0 \qquad f''(x) = 0 \Longrightarrow x = \pm 1$$

Marks: 0, 3, 5, 8

(c) Calculate the equation of the tangent at $x = 1$ and calculate the x-intercept.

$$f'(1) = -e^{-1/2} \Longrightarrow y = e^{-1/2}(-x + 2)$$

x-intercept
Marks: 0, 2, 4, 7

Section B	Contexts and Applications	150 Marks

Answer **all three** questions from this section.

Question 7 (50 Marks)

Padraig wants to recharge his mobile phone. The battery is completely empty at the moment. The model for charging the battery is given by the following equation

$$C(t) = C_\infty \left(1 - e^{kt}\right)$$

where

- $C(t)$ is the percentage of battery power available at time t,
- C_∞ is the percentage of battery power available when the battery is fully charged,
- t is the time in minutes.

(a) Padraig starts charging his phone at 19.00. Half an hour later, 23% of the battery is available.

(i) What is the value of C_∞? What can you say about the sign of k (justify your answer)?

$$C_\infty = 100\% \qquad k < 0$$

(ii) Using the information provided, calculate the value for k and give an expression for the percentage of battery available in the phone. Give your result correct to 5 decimal places.

$$23 = 100\left(1 - e^{30k}\right) \implies e^{30k} = \frac{77}{100} \implies 30k = \frac{1}{30}\ln\left(\frac{77}{100}\right) = -0.00871$$

Marks: 0, 2, 5

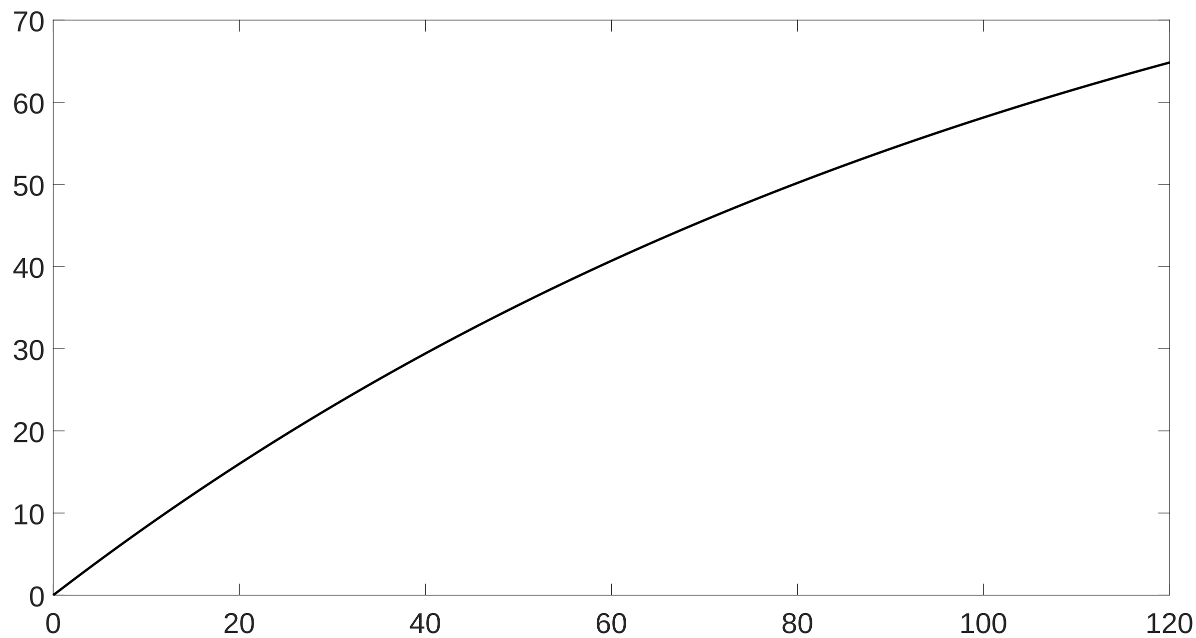
(b) Graph and analysis

(i) Plot the percentage of power available in the battery for $t \in [0 : 120]$. Draw your own x and y axis.

t	0	10	20	30	40	50	60	70	80	90	100	110	120
C	0	8.34	15.99	23	29.42	35.3	40.7	45.64	50.18	54.34	58.15	61.6	64.84

Marks: 0, 4, 7, 10, 15

(ii) Using your graph, estimate when the percentage of battery available will reach 40%. Explain how you read your result on the graph.
approximately 60 minutes



(iii) Calculate when 40% of battery is available using the formula for the percentage of battery available. Give your result correct to the nearest minute.

$$40 = 100(1 - e^{-0.00871t} \Longrightarrow e^{-0.00871t} = 0.6 \Longrightarrow t = \frac{\ln 0.6}{-0.00871} = 59$$

Marks: 0, 2, 5

(iv) How long will it take for the battery to reach a 99.5% charge? Give your result correct to the nearest minute.

$$99.5 = 100(1 - e^{-0.00871t} \Longrightarrow e^{-0.00871t} = 0.005 \Longrightarrow t = \frac{\ln 0.005}{-0.00871} = 608 \approx 10hrs$$

Marks: 0, 2, 5

(v) Do you think this result is realistic?
No

(vi) On the graph above, plot a charging curve corresponding to the phone recharging faster. Label each curve clearly. What could be a possible value of k?
k=-0.01
Marks: 0, 2, 5

(c) Rate of change

(i) Find the rates of change of the function C(t) after 1 minute and 10 minutes.

$$\frac{dC}{dt} = -kC_\infty e^{kt} \qquad \frac{dC}{dt}(1) = 0.8637 \qquad \frac{dC}{dt}(10) = 0.7985$$

Marks: 0, 2, 5

(ii) Show that the rate of change will always decrease over time.

$$\frac{d^2C}{dt^2} = -k^2 C_\infty e^{kt} < 0$$

so dC/dt always decreases.
Marks: 0, 4, 7, 10

Question 8 (50 Marks)

Paul wants to buy a new house for €250,000. He can pay a deposit of €50,000 and he needs to borrow the remaining from a bank over 20 years.

(a) Interest rate

(i) One bank offers Paul an interest rate of 0.25% per month. Calculate the annual equivalent interest rate correct to 5 decimal places.

$$AER = (1 + 0.0025)^{12} - 1 = 0.03041$$

Marks: 0, 2, 5

(ii) Another bank offers Paul a rate of 35% over 10 years. Calculate the annual equivalent interest rate correct to 5 decimal places. What should Paul do?

$$(1 + r)^{10} = 1 + 0.35 \Longrightarrow r = (1 + 0.35)^{1/10} - 1 = 0.03046$$

Marks: 0, 2, 5

(b) Paul finally goes for a loan with interest rates at r=0.2% per month. He will make his first payment of value A in one month time

(i) Calculate the present value of the first payment,

$$P = \frac{A}{1 + r}$$

Marks: 0, 2, 5

(ii) Using **induction**, show that the sum of all the present values of monthly payments is

$$PV = A\frac{(1 + r)^n - 1}{r(1 + r)^n}$$

True for n=1 as

$$PV = A\frac{(1 + r)^1 - 1}{r(1 + r)^1} = A\frac{r}{r(1 + r)} = \frac{A}{1 + r}$$

Assume true at level n

$$PV_n = (1 + r)^n$$

then

$$\begin{aligned} PV_{n+1} &= PV_n + \frac{A}{(1+r)^{n+1}} \\ &= A\frac{(1+r)^n - 1}{r(1+r)^n} + \frac{A}{(1+r)^{n+1}} \\ &= \frac{A}{(1+r)^n}\left(\frac{(1+r)^n - 1}{r} + \frac{1}{1+r}\right) \\ &= A\frac{(1+r)^{n+1} - 1}{r(1+r)^{n+1}} \end{aligned}$$

Then if true at level n, it is true at n+1. It is true for n=1 so true for $n \geq 1$
Marks: 0, 4, 7, 10

(iii) Calculate how much Paul will pay monthly to the nearest €.

$$200000 = A\frac{1.002^{240} - 1}{0.002(1.002)^{240}} \implies A = 1050$$

Marks: 0, 2, 5

(iv) How much will Paul pay back in total.

$$Total = 1045 \times 240 = 250800$$

(v) Why is this figure different from the loan amount?
Interest rates

Marks: 0, 2, 5

(c) Selling the house

(i) Calculate how much of the loan Paul will have paid back in 5 years assuming that his monthly payment is €1050. Calculate this result in present value terms correct to the nearest euro.

$$PV = 1050\frac{(1.002)^{60} - 1}{0.002(1.002)^{60}} = 59311$$

(ii) Paul wants to sell the house in five years. In present value terms, how much does he still have to pay back to the bank?

$$L = 200000 - 59311 = 140689$$

Marks: 0, 2, 5

(iii) After selling the house and repaying the mortgage, Paul wants to have €90,000 left plus his deposit. Calculate the corresponding annual increase of his house price correct to 4 decimal places.

$$P = 140689 + 50000 + 90000 = 280689$$
$$(1+r)^5 = \frac{280689}{250000} \implies r = \left(\frac{280689}{250000}\right)^{1/5} - 1 = 0.02343$$

Marks: 0, 4, 7, 10

Question 9 (50 Marks)

(a) Number of cows

(i) A dairy farmer has x cows. Each cow produces an income of 20 euros per day. It costs the farmer $x^2 - 120x + 2100$ euros everyday to house and feed the cows. Show that the daily profit is

$$\pi(x) = -x^2 + 140x - 2100 \ .$$

Justify your calculation.

$$\pi(x) = 20x - (x^2 - 120x + 2100) = -x^2 + 140x - 2100$$

Marks: 0, 2, 5

(ii) What number of cows maximises this profit?

$$\frac{d\pi}{dt} = -2x + 140 \Longrightarrow \frac{d\pi}{dt} = 0 \Longrightarrow x = 70$$

Marks: 0, 2, 5

(iii) Calculate the maximum profit and the break even points.

$$\pi(70) = 2800$$
$$\pi(x) = 0 \Longrightarrow x = \frac{140 \pm \sqrt{140^2 - 8400}}{2} = 70 \pm 52.91$$

break even points: 18 and 122 cows
Marks: 0, 4, 57, 10

(b) The amount of milk produced by each cow varies with time. The number of litres of milk may be approximated using the function

$$f(t) = 20 - 4\cos\left(\frac{2\pi}{366}t\right)$$

where t is the number of days from (and including) the first of January and the ratio $2\pi t/366$ is expressed in **radians**. Note this is a **leap** year.

(i) How many litres of milk does a cow produce on 1 May?

$$t = 31 + 29 + 31 + 31 + 30 + 1 = 122$$
$$f(122) = 20 - 4\cos\left(\frac{2\pi}{366}122\right) = 22$$

Marks: 0, 2, 5

(ii) Calculate the first order derivative of function f.

$$f'(t) = \frac{8\pi}{366}\sin\left(\frac{2\pi}{366}t\right)$$

Marks: 0, 2, 5

(iii) When do cows produce the maximum and minimum amount of milk? What are these amounts?

$$
\begin{aligned}
& f'(t) = 0 \Longrightarrow \sin\left(\frac{2\pi}{366}t\right) = 0 \\
& \Longrightarrow \frac{2\pi}{366}t = \pi \quad or \quad \frac{2\pi}{366}t = 2\pi \\
& \Longrightarrow t = 183 \quad or \quad t = 366
\end{aligned}
$$

Two dates: 1 July, 31 December

Marks: 0, 4, 7, 10

(iv) How many litres of milk does a cow produce on average between 1 January and 31 March?

$$
\begin{aligned}
I &= \frac{1}{91}\int_0^{91}\left(20 - 4\cos\left(\frac{2\pi}{366}t\right)\right) \\
&= \frac{1}{91}\left[20t - \frac{366 \times 4}{2\pi}\sin\left(\frac{2\pi}{366}t\right)\right]_0^{91} \\
&= \frac{1}{91}\left[20(91) - \frac{366 \times 4}{2\pi}\sin\left(\frac{2\pi}{366}91\right) - 0\right] \\
&= 17.44
\end{aligned}
$$

Marks: 0, 4, 7, 10