Leaving Certificate Examination, 2017

Sample paper prepared by Leamy Maths Community

Mathematics

Paper 1

Higher Level

Sunday 23 April

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Leamy Maths Community

Solutions

300 marks

Sample Instructions

There are two sections in this examination paper:

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer questions as follows:

In Section A, answer all six questions.
In Section B, answer all two questions.

Write your answers in the spaces provided in this booklet. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the booklet of *Formulae and Tables*. You must return it at the end of the examination.

You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A — Concepts and Skills — 150 Marks

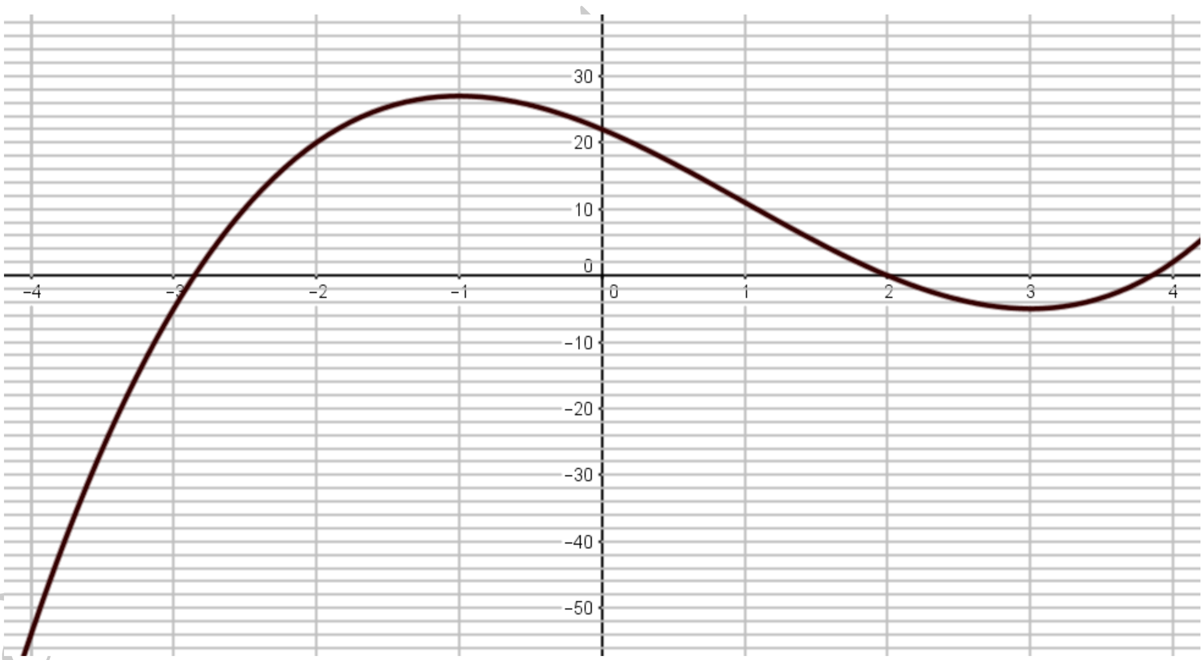
Answer **all six** questions from this section.

Question 1 (25 Marks)

(a) Plot the function $f(x)$ on the attached graph.

$$f(x) = x^3 - 3x^2 - 9x + 22$$

-4	-3	-1	0	1	3	4
-54	-5	27	22	11	-5	2



Marks: 0, 3, 5, 7

(b) Identify one root on the graph and calculate the other 2 roots. Express the roots in the format $(a + b\sqrt{c})/d$

From graph, x=2 is a solution
So $(x-2)$ is the factor.

$$\begin{array}{rl} & \quad\; x^2 \quad - x - 11 \\ x-2\, \big) & \overline{x^3 - 3x^2 \quad - 9x + 22} \\ & \underline{-x^3 + 2x^2} \\ & \qquad -x^2 \quad - 9x \\ & \qquad \underline{\;\; x^2 \quad - 2x} \\ & \qquad\qquad -11x + 22 \\ & \qquad\qquad \underline{\;\; 11x - 22} \\ & \qquad\qquad\qquad\quad 0 \end{array}$$

$$x = 2 \qquad x = \frac{1 - 3\sqrt{5}}{2} \approx -2.8541 \qquad x = \frac{1 + 3\sqrt{5}}{2} \approx 3.3541$$

Marks: 0, 3, 5, 7, 10

(c) Calculate the coordinates of the turning points and place the points on the graph
Turning Points - Let $f'(x) = 0$ and solve

$$\begin{aligned} f'(x) = 3x^2 - 6x - 9 &= 0 \\ 3(x+1)(x-3) &= 0 \\ x = -1 \quad x &= 3 \end{aligned}$$

Fill x values into $f(x)$ to find corresponding y values
Turning points $(-1, 27)$, $(3, -5)$.

Marks: 0, 3, 5, 7

Question 2 (25 Marks)

Consider the three complex numbers

$$z_1 = 1 + 2i \qquad z_2 = -1 - i \qquad z_3 = 1 + 2\sqrt{3} + i\left(2 - \sqrt{3}\right)$$

(a) Identify the 3 numbers z_1, z_2 and z_3 on the graph. Justify your answer.
Marks: 0, 3, 5

(b) Calculate the complex number z_4 representing the point z_2 after a clockwise 90 degrees rotation and a factor 3 enlargement. Place the point of the graph and calculate $|z_4 - z_2|$.
90 degrees clockwise rotation - Multiply by $-i$
Factor of 3 enlargement - Multiply by 3
So in total, multiply z_2 by $-3i$ to calculate z_4

$$\begin{aligned} z_4 = -3iz_2 &= 3i - 3 \\ |z_4 - z_2| = |-2 + 4i| &= 2\sqrt{5} \end{aligned}$$

Marks: 0, 3, 5, 7, 10

(c) Calculate the ratio z_3/z_1. Calculate the modulus and argument of this ratio

$$\begin{aligned}\frac{z_3}{z_1} &= \frac{1+2\sqrt{3}+i\left(2-\sqrt{3}\right)}{1+2i} \times \left(\frac{1-2i}{1-2i}\right) \\ &= \frac{1+2\sqrt{3}-2i-4\sqrt{3}i+2i-\sqrt{3}i-4i^2+2\sqrt{3}i^2}{1-2i+2i-4i^2} \\ &= \frac{1+2\sqrt{3}-4\sqrt{3}i-\sqrt{3}i+4-2\sqrt{3}}{1+4} \\ &= \frac{5-5\sqrt{3}i}{5} \\ &= 1-\sqrt{3}i\end{aligned}$$

$$\begin{aligned}\frac{z_3}{z_1} &= 1-i\sqrt{3} \\ \left|\frac{z_3}{z_1}\right| &= 2 \\ Arg\left(\frac{z_3}{z_1}\right) &= -\frac{\pi}{3}rad = -60^\circ \\ \text{OR} \qquad Arg\left(\frac{z_3}{z_1}\right) &= \frac{5\pi}{3} = 300^\circ\end{aligned}$$

Marks: 0, 3, 5, 7, 10

Question 3 (25 Marks)

The population of Ireland was 3.63 million in 1996 and 4.58 million in 2011. The population can be modelled with an exponential function,

$$P(t) = Ae^{b\times t}$$

where A is in million, t is in years and t=0 corresponds to 1996.

(a) Calculate A and b. Use four significant figures.

$$\begin{aligned}P(t) &= 3.63e^{bt} \\ P = 4.58 \quad t &= 15 \\ 4.58 &= 3.63e^{b(15)} \\ \frac{4.58}{3.63} &= e^{15b} \\ \ln\frac{4.58}{3.63} &= 15b \\ 0.155 &= b \\ P(t) &= 3.63e^{0.0155t}\end{aligned}$$

Marks: 0, 3, 5, 7, 10

(b) What will be the population in 2017? Give your answer with 4 significant digits.

$$t = 21$$
$$P(21) = 3.63e^{0.0155(21)}$$
$$P(21) = 5.0265\ million$$

Marks: 0, 3, 5

(c) What is the rate of change of the population in 2020?
Rate of change - Differentiate

$$P'(t) = 0.0155 \times 3.63e^{0.0155t}$$
$$P'(24) = 0.0155 \times 3.63e^{0.0155\times 24}$$
$$P'(24) = 0.0816$$

The rate of change is 81,600 people per year
Marks: 0, 3, 5

(d) In what year will the population reach 5.5 million?

$$5.5 = 3.63e^{0.0155t}$$
$$\frac{5.5}{3.63} = e^{0.0155t}$$
$$\ln\frac{5.5}{3.63} = 0.0155t$$
$$t = \frac{1}{0.0155}\ln\left(\frac{5.5}{3.63}\right)$$
$$t \approx 26.8$$

Will reach 5.5 million in 2022 Marks: 0, 3, 5

Question 4 (25 Marks)

Consider the function

$$f(x) = e^{x\sin x} \qquad \sin(-x) = -sinx$$

(a) Where is the function defined? Show that $f(x) = f(-x)$.

Function defined in $\mathbb{R}$.

$$f(-x) = e^{-x\sin(-x)}$$
$$= e^{(-x)(-\sin x)}$$
$$= e^{x\sin x}$$
$$f(-x) = f(x)$$

Marks: 0, 3, 5

(b) Show that for $x \geq 0$

$$e^{-x} \leq f(x) \leq e^{x}$$

$$\begin{aligned}
&\Longrightarrow e^{-x} \leq f(x) \leq e^{x} \qquad x > 0 \\
&\Longrightarrow e^{-x} \leq e^{x \sin x} \leq e^{x} \qquad x > 0 \\
&\Longrightarrow \ln\left(e^{-x}\right) \leq \ln\left(e^{x \sin x}\right) \leq \ln\left(e^{x}\right) \qquad x > 0 \\
&\Longrightarrow -x \leq x \sin x \leq x \qquad x > 0 \\
&\Longrightarrow -1 \leq \sin x \leq 1 \quad \text{Which we know to be true}
\end{aligned}$$

Marks: 0, 3, 5, 7

(c) Show that the first order derivative of $g(x) = x \sin x$ is $g'(x) = \sin x + x \cos x$. Hence calculate the first order derivative of f

$$\begin{aligned}
g(x) &= x \sin x \\
u = x \quad v &= \sin x \\
\frac{du}{dx} = 1 \quad \frac{dv}{dx} &= \cos x \\
\frac{dy}{dx} &= x \cos x + \sin x
\end{aligned}$$

$$\begin{aligned}
f(x) &= e^{x \sin x} \\
&\quad \text{Differentiate with Chain Rule} \\
f'(x) &= e^{x \sin x}(\sin x + x \cos x)
\end{aligned}$$

Marks: 0, 3, 5, 8

(d) Prove that turning points are solutions of the equation

$$x = -\tan(x)$$

and show that x=0 is a turning point.

$$\begin{aligned}
f'(x) &= 0 \\
e^{x \sin x}(\sin x + x \cos x) &= 0 \\
e^{x \sin x} = 0 \quad (\sin x + x \cos x) &= 0 \\
\sin x + x \cos x &= 0 \\
x \cos x &= -\sin x \\
x &= -\frac{\sin x}{\cos x} \\
x &= -\tan x
\end{aligned}$$

Marks: 0, 3, 5

Question 5 (25 Marks)

Mark is setting up a retirement fund for his pension. The pension scheme is designed in such a way that he will receive annual payments of €40,000 for 30 years. He starts saving for his pension 35 years before retiring.

(a) Mark has to choose between an annual rate of 7% or a monthly rate of 0.5%. Calculate the corresponding annual rate (use two digit precision for the % value). What solution should Mark prefer?

$$r_{annual} = \left(1 + \frac{0.5}{100}\right)^{12} = 1.0617 = 6.17\%$$

Marks: 0, 3, 5

(b) How much money needs to be in the retirement fund on the day that Mark retires in order to fund the 30 annual payments of €40,000 with payments made at the beginning of the year.? Round up your result to the nearest euro. Take an annual rate of 7%. Justify your answer in terms of present value.
Series:

$$40,000 + \frac{40000}{1.07} + \frac{40000}{(1.07)^2} + \text{.........} + \frac{40000}{(1.07)^{29}}$$

$$S_{30} = \frac{40000(1 - \frac{1}{1.07}^{30})}{1 - \frac{1}{1.07}}$$

$$\approx € \, 531,107$$

Marks: 0, 3, 5, 7, 10

(c) How much should Mark deposit annually at the beginning of each year to ensure that he has adequate funds in his retirement account on the day he retires? Justify the answer in terms of future value and round up your results to the nearest euro. Payments are made at the beginning of every year. Take an annual rate of 7%.

$$P(1+0.07)^{35} + P(1+0.07)^{34} + \cdots + P(1+0.07) = 531107$$

$$P[(1.07)^{35} + (1.07)^{34} + \cdots + (1.07)] = 531107$$

$$P\left(\frac{1.07(1 - 1.07^{3}5)}{1 - 1.07}\right) = 531107$$

$$P(147.9134598) = 531107$$

$$P \approx €3591$$

Marks: 0, 3, 5, 7, 10

Question 6 (25 Marks)

Consider the series

$$u_n = 6^n + 4$$

(a) Calculate u_0, u_1, u_2, u_3, u_4

$$u_0 = 5 \quad u_1 = 10 \quad u_2 = 40 \quad u_3 = 220 \quad u_4 = 1300$$

Marks: 0, 3, 5

(b) Show that for $n \geq 1$, u_n is even.

$$u_n = 6^n + 4 = 6 \times 6^{n-1} + 4 = 2\left(3 \times 6^{n-1} + 2\right)$$

If $n \geq 1$, 6^{n-1} is an integer so $3 \times 6^{n-1} + 2$ is an integer, so u_n is even. Marks: 0, 3, 5

(c) Prove by induction that u_n can always be divided by 5

- Property P(n): $6^n + 4$ can be divided by 5
- $u_0 = 5$ may be divided by 5
- Assume P(n) is true: $6^n + 4$ can be divided by 5

$$\begin{aligned} 6^n + 4 &= 5Z \\ 6^n &= 5Z - 4 \end{aligned}$$

- What can be said about $6^{n+1} + 4$?

$$\begin{aligned} 6^{n+1} + 4 &= 6 \times 6^n + 4 \\ &= 6 \times (5Z - 4) + 4 \\ &= 30Z - 24 + 4 \\ &= 30Z - 20 \\ &= 5(6Z - 4) \end{aligned}$$

  $6^n + 4$ can be divided by 5, this is the assumption. 30Z -20 can be divided by 5 as well so all terms can be divided by, this means that if $6^n + 4$ can be divided by 5, then $6^{n+1} + 4$ can be divided by 5.

- If P(n) is true, then P(n+1) is true. Property P(n) is inductive, it is true for n=0, so the property is true for $n \geq 0$

Marks: 0, 5, 8, 10, 15

Section B	Contexts and Applications	150 Marks

Answer **all three** questions from this section.

Question 7 (50 Marks)

Aoife wants to build the model of a bridge. She first needs to draw a cross section on the attached graph. The body of the bridge is rectangular shaped with corners with coordinates$(-6, 0)$, $(-6, 4)$, $(6, 4)$, $(6, 0)$. The arch of the bridge is delimited by a function of the form

$$f(x) = ax^2 + bx + c$$

All values are in cm.

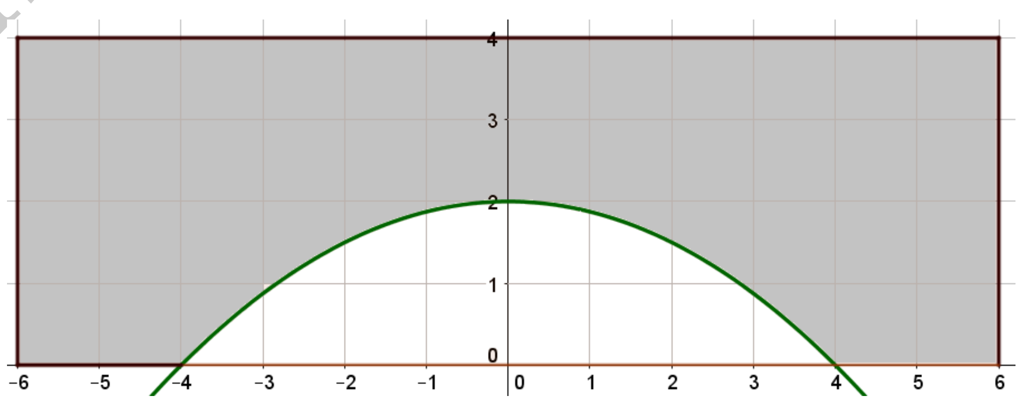
(a) The arch of the bridge passes through point $(-4, 0)$, and has a turning point at $(0, 2)$. Solve for a, b and c to show that the arch of the bridge is delimited by the function

$$f(x) = 2 - \frac{x^2}{8}$$

$$\begin{aligned}
(-4, 0) \quad & \implies f(-4) = 0 \\
& \implies f(-4) = 16a - 4b + c = 0 \\
(0, 2) \quad & \implies f(0) = 2 \\
& \implies f(0) = 0a + 0b + c = 2 \\
& \implies c = 2 \\
(0, 2) \quad \text{is a turning point} & \implies f'(0) = 0 \\
& \implies f'(0) = 2a(0) + b = 0 \\
& \implies b = 0 \\
& \implies c = 2,\ b = 0, a = -\frac{1}{8} \\
& \implies f(x) = 2 - \frac{x^2}{8}
\end{aligned}$$

Marks: 0, 3, 5, 7, 10

(b) Hence draw the bridge on the graph below [5 points].



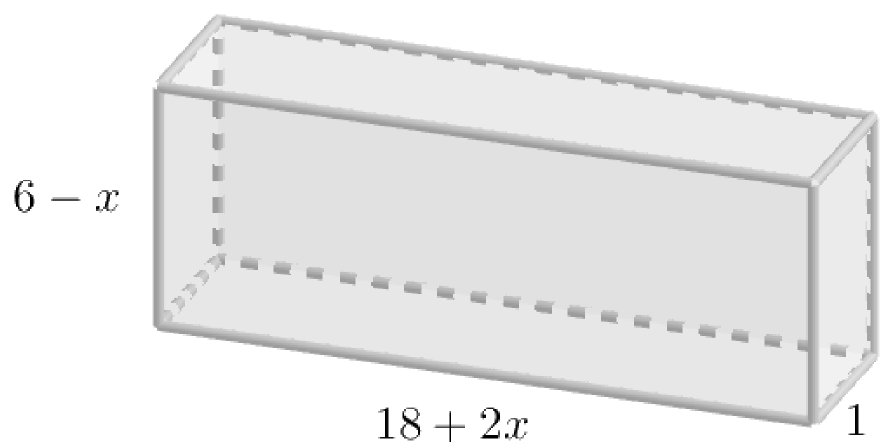
Marks: 0, 3, 5

(c) Aoife wants to build a 3 dimensional model of the bridge. All faces will be painted with a very inexpensive material. However, she plans to use an expensive golden paint on one of the cross sections of the bridge. The paint costs €25 per square centimetre. Calculate the area of the cross section and then specify how much the paint will cost. Give all results as a fraction and then approximate the cost value to the nearest euro.

$$\begin{aligned} Area_{Arch} &= \int_{-4}^{4}\left(2-\frac{x^2}{8}\right)dx \\ &= \left[2x-\frac{x^3}{24}\right]_{-4}^{4} \\ &= 8-\frac{64}{24}-\left(-8+\frac{64}{24}\right) \\ &= \frac{32}{3} \\ Area_{Xsection} &= 4\times 12-\frac{32}{3}=\frac{112}{3} \\ Cost &= 25\times\frac{112}{3}=\frac{2800}{3}\approx €933 \end{aligned}$$

Marks: 0, 5, 8, 10, 15

(d) Since the paint is so expensive, Aoife would like to put her model in a plexiglas casing with a parallelepiped shape. For aesthetic reasons, she would like the case to be 1cm deep, (18+2x) cm long and (6-x) cm wide. Calculate the area of plexiglas necessary to build the case.



$$\begin{aligned} SurfaceArea &= 2\times[(6-x)(1)+(18+2x)(1)+(6-x)(18+2x)] \\ &= 2\times\left[24+x+108-6x-2x^2\right] \\ &= 264-10x-4x^2 \end{aligned}$$

Marks: 0, 3, 5

(e) Hence find the size of case that maximises the area of plexiglas necessary. Calculate the area in square cm.

$$
\begin{aligned}
f'(x) &= -8x - 10 \\
-8x - 10 &= 0 \\
-8x &= 10 \\
x &= -1.25
\end{aligned}
$$

Surface Area is $15.5 \times 7.25 \times 1$ cm.
Total area 270.25cm^2

Marks: 0, 3, 5

(f) Alternatively, Aoife could select the value of x that maximises the volumes enclosed in the plexiglas. Calculate this value and the corresponding area of plexiglas necessary to build the case.

$$
\begin{aligned}
V(x) &= (18 + 2x) \times (6 - x) \times 1 = 108 - 6x - 2x^2 \\
V'(x) &= -6 - 4x \Longrightarrow x = -1.5
\end{aligned}
$$

Volume $=112.5\text{cm}^3$, Area$=270$ cm^2

Marks: 0, 3, 5, 7, 10

Question 8 (45 Marks)

In a lab, water droplets are simulated. They are produced by introducing small spheres of radius $r =$ mm in a very moist environment. The volume of the sphere increases at a rate of $\pi/2$ mm^3/s

(a) Find the rate of change of the radius as a function of r expressed in mm and calculate the value at $r = 1$mm.

$$
\frac{dV}{dt} = \frac{\pi}{2} \qquad \frac{dr}{dt} = ?
$$

$$
V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2
$$

$$
\begin{aligned}
\frac{dr}{dt} &= \frac{dV}{dt} \times \frac{dr}{dV} \\
\frac{dr}{dt} &= \frac{\pi}{2} \times \frac{1}{4\pi r^2} \\
\frac{dr}{dt} &= \frac{\pi}{8\pi r^2} \\
\frac{dr}{dt} &= \frac{1}{8r^2}
\end{aligned}
$$

For $r = 1\text{mm}$ this leads to

$$\frac{dr}{dt} = 0.125 \text{ mm/s}$$

Marks: 0, 3, 5, 7, 10

(b) Find the rate of change of the surface area when $r = 2\text{mm}$

$$\begin{aligned} \frac{dA}{dt} &= \frac{dA}{dr}\frac{dr}{dt} \\ &= 8\pi r \times \frac{1}{8r^2} = \frac{\pi}{r} \end{aligned}$$

For $r = 2\text{mm}$, this leads to

$$\frac{dA}{dt} = \frac{\pi}{2} = 1.57\text{mm}^2/\text{s}$$

Marks: 0, 3, 5, 7, 10

(c) The conditions are such that the volume of the sphere actually varies as

$$V(t) = 20\left(1 - e^{-0.1t}\right)$$

where V is in mm^3.

(i) According to this model, what is the maximum volume that can be reached by the droplet? Hint: Consider the function as $t \to \infty$

$$V(t) = 20\left(1 - e^{-0.1t}\right)$$

As $t \to \infty$

$$\begin{aligned} V &= 20(1 - e^{-0.1(\infty)}) \\ &= 20(1 - e^{-\infty}) \\ &= 20(1 - 0) \qquad \text{Since } e^{-\infty} = 0 \\ &= 20 \end{aligned}$$

When t grows very large, the exponential becomes very small and the limit of V is $V = 20\text{mm}^3$.

Marks: 0, 3, 5

(ii) Calculate the time when half of this volume is reached. Give your result with two significant digits.

$$\begin{aligned} 10 = 20\left(1 - e^{-0.1t}\right) &\implies 1 - e^{-0.1t} = \frac{1}{2} \\ &\implies e^{-0.1t} = \frac{1}{2} \\ &\implies 0.1t = \ln 2 \\ &\implies t = \frac{\ln 2}{0.1} = 6.93s \end{aligned}$$

Marks: 0, 3, 5, 7, 10

(iii) Calculate the rate of change of the volume. At what time will the volume vary at the rate of $\pi/2$ mm$^3/s$ used in the first part? Give your result with two significant digits.
Rate of change

$$\begin{aligned} V &= 20\left(1 - e^{-0.1t}\right) \\ V &= 20 - 20e^{-0.1t} \\ \frac{dV}{dt} &= -20e^{-0.1t} \times -0.1 \\ \frac{dV}{dt} &= 2e^{-0.1t} \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt} = \frac{\pi}{2} &\implies 2e^{-0.1t} = \frac{\pi}{2} \\ &\implies -0.1t = \ln\left(\frac{\pi}{4}\right) \\ &\implies t = -\frac{\ln\left(\frac{\pi}{4}\right)}{-0.1} = 2.42s \end{aligned}$$

Marks: 0, 3, 5, 7, 10

Question 9 (55 Marks)

(a) Given that

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

show that

$$\int_0^1 \frac{dx}{1 + x^2} = \frac{\pi}{4}$$

$$\begin{aligned} \int_0^1 \frac{dx}{1 + x^2} &= \int_0^1 \frac{1}{x^2 + 1} dx \\ &= \frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\Big|_0^1 \\ &= \frac{1}{1}\tan^{-1}\left(\frac{1}{1}\right) - \frac{1}{1}\tan^{-1}\left(\frac{0}{1}\right) \\ &= \frac{\pi}{4} - 0 \\ &= \frac{\pi}{4} \end{aligned}$$

Marks: 0, 3

(b) Approximate the integral: show that

$$\left(1+x^2\right)\left(1-x^2+x^4-x^6+x^8\right)=1+x^{10}$$

hence show that

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-\frac{x^{10}}{1+x^2}$$

$$\begin{aligned}\left(1+x^2\right)\left(1-x^2+x^4-x^6+x^8\right)&=1-x^2+x^4-x^6+x^8+x^2-x^4+x^6-x^8+x^{10}\\&=1+x^{10}\end{aligned}$$

$$\begin{aligned}\left(1+x^2\right)\left(1-x^2+x^4-x^6+x^8\right)&=1+x^{10}\\\left(1-x^2+x^4-x^6+x^8\right)&=\frac{1+x^{10}}{1+x^2}\\\left(1-x^2+x^4-x^6+x^8\right)&=\frac{1}{1+x^2}+\frac{x^{10}}{1+x^2}\\1-x^2+x^4-x^6+x^8-\frac{x^{10}}{1+x^2}&=\frac{1}{1+x^2}\end{aligned}$$

Marks: 0, 3, 5, 7

(c) Using the results of part (b) show that

$$\int_0^1\frac{dx}{1+x^2}=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right)-err(x)\qquad\text{where}\qquad err(x)=\int_0^1\frac{x^{10}}{1+x^2}$$

$$\begin{aligned}\int_0^1\frac{dx}{1+x^2}&=\int_0^1\left(1-x^2+x^4-x^6+x^8-\frac{x^{10}}{1+x^2}\right)dx\\&=\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}\right]_0^1-\int_0^1\frac{x^{10}}{1+x^2}dx\\&=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right)-\int_0^1\frac{x^{10}}{1+x^2}dx\end{aligned}$$

Marks: 0, 3, 5, 7, 10

(d) Since

$$\int_0^1\frac{dx}{1+x^2}=\frac{\pi}{4}$$

prove that

$$\pi=4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-err(x)\right)$$

and evaluate the approximation of π with 3 digit precision assuming the $err(x) \approx 0$.

$$\begin{aligned}
\pi &= 4\int_0^1 \frac{dx}{1+x^2} \\
&= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - err(x)\right) \\
\pi &\approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right) \approx 3.334
\end{aligned}$$

Marks: 0, 3, 5, 7, 10

(e) The error term is between the upper and lower values

$$\int_0^1 \frac{x^{10}}{2} dx \le err(x) \le \int_0^1 x^{10} dx$$

(You are not required to show this). Calculate the upper and lower values of this inequality given by the integrals. Using the formula in part (d), give the corresponding interval including π

$$\begin{aligned}
&\Longrightarrow \int_0^1 \frac{x^{10}}{2} dx \le err(x) \le \int_0^1 x^{10} dx \\
&\Longrightarrow \frac{x^{11}}{22}\Big|_0^1 \le err(x) \le \frac{x^{11}}{11}\Big|_0^1 \\
&\Longrightarrow \frac{1}{22} \le err(x) \le \frac{1}{11} \\
&\Longrightarrow \frac{2}{11} \le err(x) \le \frac{4}{11} \\
&\Longrightarrow -\frac{4}{11} \le -err(x) \le -\frac{2}{11} \\
&\Longrightarrow 3.334 - \frac{4}{11} \le 3.334 - err(x) \le 3.334 - \frac{2}{11} \\
&\Longrightarrow 2.970 \le \pi \le 3.152
\end{aligned}$$

Marks: 0, 3, 5, 7, 10

(f) Generalising the method, it is possible to show that the difference between the upper and lower estimation of is

$$Diff(n) = \frac{2}{2n+1}$$

where n is the number of terms used in the sum (you are not required to prove this). How many terms are required to calculate π with a difference between the upper and lower estimation lower than 0.0001. Is it possible to do this estimation manually?

$$\begin{aligned} & Diff(n) \leq 0.0001 \\ \implies & \frac{2}{2n+1} \leq 0.0001 \\ \implies & \frac{2}{0.0001} \leq 2n+1 \\ \implies & \frac{2}{0.0001} \leq 2n+1 \\ \implies & 19999 \leq 2n \\ \implies & 9999.5 \leq n \end{aligned}$$

You need 10000 terms, impossible to calculate manually.

Marks: 0, 3, 5

(g) Another approximation of π was developed in the middle age

$$\pi = \sqrt{12}\sum_{k=0}^{n}\left(\frac{\left(-\frac{1}{3}\right)^k}{2k+1}\right)$$

- Calculate the value for $n = 0$, $n = 1$, $n = 2$, $n = 3$ accurate to 5 digits

$$u_0 = 3.46410 \quad u_1 = 3.07920 \quad u_2 = 3.15618 \quad u_3 = 3.13785$$

Marks: 0, 3, 5

- A study of the series shows that for n=3, the error committed when using the first 3 terms of the sum verifies

$$err(3) \leq \sum_{k=4}^{\infty}\frac{\sqrt{12}}{3}\frac{1}{3^k}$$

(you are not required to prove this formula). Show that Calculate the maximum value for this error and the corresponding upper and lower approximation for π

$$\frac{\sqrt{12}}{3}\sum_{k=4}^{\infty}\frac{1}{3^k} = \frac{\sqrt{12}}{3}\times\frac{1}{3^4}\times\frac{1}{1-\frac{1}{3}} = \frac{\sqrt{12}}{3}\times\frac{1}{81}\times\frac{3}{2} = 0.02138$$

$$3.13785 - 0.02138 \leq \pi \leq 3.13785 + 0.02138 \implies 3.116466657 \leq \pi \leq 3.159233343$$

Marks: 0, 3, 5