Leaving Certificate Examination, 2017

Sample paper prepared by Leamy Maths Community

Mathematics

Paper 2

Higher Level

Monday 1 May

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Solutions

300 marks

Sample Instructions

There are two sections in this examination paper:

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	2 questions

Answer questions as follows:

In Section A, answer all six questions.
In Section B, answer both questions.

Write your answers in the spaces provided in this booklet. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the booklet of *Formulae and Tables*. You must return it at the end of the examination.

You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	150 Marks

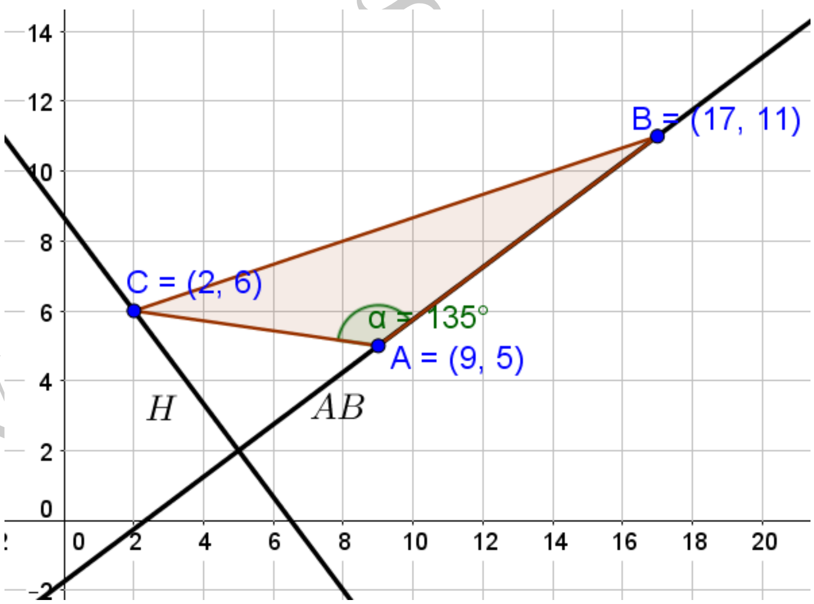
Answer **all six** questions from this section.

Question 1 (25 Marks)

(a) Calculate the equation of the line defined by points $A(9,5)$ and $B(17,11)$ and plot it on the graph.

$$m = \frac{11-5}{17-9} = \frac{3}{4}$$
$$y - y_1 = m(x - x_1)$$
$$y - 5 = \frac{3}{4}(x - 9)$$
$$3x - 4y - 7 = 0$$

[5 marks]



(b) Calculate the equation of line H that is perpendicular to (AB) and passes point $C(2,6)$. Draw the line on the graph. [5 marks]

$$\frac{3}{4} \perp -\frac{4}{3}$$
$$y - 6 = -\frac{4}{3}(x - 2)$$
$$4x + 3y - 26 = 0$$

(c) Calculate the area of triangle ABC. Draw the triangle on the graph [10 marks]

$$\begin{aligned} &(9,5) \quad (17,11) \quad (2,6) \\ &(7,-1) \quad (15,5) \quad (0,0) \\ &Area = \frac{1}{2}|x_1y_2 - x_2y_1| \\ &Area = \frac{1}{2}|(7)(5) - (-1)(15)| \\ &Area = 25 \end{aligned}$$

(d) Calculate the angle $< A$. Give your results in degrees with two significant digits. [5 marks]

Calculate lengths of triangle ABC $\quad |AB| = 10, \quad |AC| = 5\sqrt{2}, \quad |BC| = 5\sqrt{10}$

Cosine Rule $\quad \cos < BAC = \dfrac{AB^2 + AC^2 - BC^2}{2AB \times AC} = \dfrac{100 + 50 - 250}{100\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$$\implies | < BAC| = \cos^{-1}(-\frac{\sqrt{2}}{2}) = 135°$$

Question 2 (25 Marks)

The equation of the two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are:

$$\begin{aligned} \mathcal{C}_1 \qquad & x^2 + y^2 - 12x - 10y + 43 = 0 \\ \mathcal{C}_2 \qquad & x^2 + y^2 - 8x - 6y + 23 = 0 \end{aligned}$$

(a) Find the centre and radius for both circles [5 marks]

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre=$(-g, -f)$
Radius=$\sqrt{g^2 + f^2 - c}$

$$\begin{aligned} \mathcal{C}_1 \qquad & \text{Centre:}A(6,5) \qquad R_1 = 3\sqrt{2} \\ \mathcal{C}_2 \qquad & \text{Centre:}B(4,3) \qquad R_2 = \sqrt{2} \end{aligned}$$

(b) Show that the two circles are touching [5 marks]

Distance between centres of circles: $\quad |AB| = \sqrt{(6-4)^2 + (5-3)^2} = 2\sqrt{2}$

$$R_2 - R_1 = 2\sqrt{2}$$

The distance between the two centres is equal to the radii subtracted. Therefore the circles are touching internally.

(c) Calculate the coordinates of the common point. [10 marks]
Subtract the circles to get the equatoin of the common tangent.

$$\begin{aligned} x^2 + y^2 - 12x - 10y + 43 &= 0 \\ -(x^2 + y^2 - 8x - 6y + 23 &= 0) \\ \hline -4x - 4y + 20 &= 0 \\ \text{Common Tangent:} \quad x + y - 5 &= 0 \end{aligned}$$

Then find point of intersection of common tanget and either circle:

$$\begin{aligned} x^2 + y^2 - 12x - 10y + 43 &= 0 \\ x + y &= 5 \\ (5-y)^2 + y^2 - 12(5-y) - 10y + 43 &= 0 \\ 25 - 10y + y^2 + y^2 - 60 + 12y - 10y + 43 &= 0 \\ 2y^2 - 8y + 8 &= 0 \\ y^2 - 4y + 4 &= 0 \\ (y-2)(y-2) &= 0 \\ y &= 2 \\ x &= 5 - 2 \\ x &= 3 \\ \text{Common Point:} \quad &= (3, 2) \end{aligned}$$

(d) Show that line $y = x + 1$ is tangent to circle $\mathcal{C}_2$. [5 marks]
Show that the perpendicular distance from the centre of the circle to the tangent is the same length as the radius.

$$\begin{aligned} \text{Centre of C2:} &= (4, 3) \\ \text{Tangent:} &\to y = x + 1 \\ &\to x - y + 1 = 0 \\ \text{Formula:} &\to \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ \text{Perpendicular distance:} &\to \frac{|(1)(4) + (-1)(3) + 1|}{\sqrt{(1)^2 + (1)^2}} \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2} \end{aligned}$$

The perpendicular distance from centre of C_2 to tangent:$y = x + 1$ is $\sqrt{2}$
The radius of C_2 is $\sqrt{2}$
Therefore $y = x + 1$ is tangent to circle $\mathcal{C}_2$.

Question 3 (25 Marks)

An industrial produces bags of sweets. There should be at least 20 sweets per bag. However, due to faults in production, in 20% of bags, there are 19 sweets or less. Paul buys 8 bags of sweets and counts how many bags have 19 sweets or less. In the following, use four significant digits for all your results

(a) Explain why this test constitutes a Bernoulli experiment. [5 marks]

- Limited number of trials
- Each trial can be split into success or failure
- Each trial is independent
- Probability of success remains the same throughout trials

(b) What is the probability that he buys exactly one bag with 19 sweets or less. [5 marks]

$$\begin{aligned} n &= 8 \\ p &= 0.2 \\ q &= 0.8 \\ r &= 1 \end{aligned}$$

$$P = C_1^8 \times 0.2 \times 0.8^7 = .3355$$

(c) What is the probability that he got three or more bags with 19 sweets or less? [10 marks]

$$\begin{aligned} P &= 1 - [P(r = 0) + P(r = 1) + P(r = 2)] \\ P &= 1 - \left[(C_0^8 \times 0.2^0 \times 0.8^8) + (C_1^8 \times 0.2 \times 0.8^7) + (C_2^8 \times 0.2^2 \times 0.8^6)\right] \\ &= 1 - (0.1678 + 0.3355 + 0.2936) = 0.2031 \end{aligned}$$

(d) Paul buys 8 bags of sweets every week during a year. He notices that half the time, he gets at least three bags with 19 sweets or less. What can you say?

[5 marks]
The percentage of bags with 19 sweets or less is probably higher than the 20% estimated by the factory.

Question 4 (25 Marks)

(a) Given two events E and F, explain what is meant by $P(E|F)$, $P(F|E)$ and and express these probabilities in terms of P(E), P(F) and $P(E \cap F)$. Find a relation between $P(E|F)$ and $P(F|E)$

[8 marks]

- $P(E|F)$ probability of event E occurring given that event F has occurred
- $P(F|E)$ probability of event E occurring given that event F has occurred

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \qquad P(F|E) = \frac{P(E \cap F)}{P(E)}$$
$$P(E|F) = P(F|E)\frac{P(E)}{P(F)}$$

(b) In a class of 30 students in a girls school,

- 20 students study higher level maths,
- 15 students study higher level French,
- 8 students study higher level maths and higher level French

(i) Are the events studying higher level maths and higher level French independent? [6 marks]

Check for Independent Events: $P(A) \times P(B) = P(A \cap B)$

$$P(M) = \frac{20}{30} = \frac{2}{3}$$
$$P(F) = \frac{15}{30} = \frac{1}{2}$$
$$P(M \cap F) = \frac{8}{30} = \frac{4}{15}$$
$$P(M) \times P(F) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$
$$\frac{4}{15} \neq \frac{1}{3}$$

Events are not independent.

In the following, give all your results as simplified fractions. A student is picked randomly. What is the probability that

(ii) She studies higher level maths or higher level French? [4 marks]

$$P(M \cup F) = P(M) + P(F) - P(M \cap F) = \frac{2}{3} + \frac{1}{2} - \frac{4}{15} = \frac{9}{10}$$

(iii) She studies higher level French given that she studies higher level maths? [4 marks]

$$P(F|M) = \frac{P(M \cap F)}{P(M)} = \frac{\frac{4}{15}}{\frac{2}{3}} = \frac{2}{5}$$

(iv) Using the previous result, calculate the probability that she studies higher level maths given that she studies higher level French. [3 marks]

$$P(M|F) = P(F|M)\frac{P(M)}{P(F)}$$
$$= \frac{2}{5} \times \frac{\frac{2}{3}}{\frac{1}{2}}$$
$$= \frac{8}{15}$$

Question 5 (25 Marks)

(a) Show that

$$\cos 3x = 4\cos^3 x - 3\cos x$$

[10 marks]

$$\begin{aligned}
\cos(3x) &= \cos(2x + x) \\
&= \cos(2x)\cos x - \sin(2x)\sin x \\
&= \left(\cos^2 x - \sin^2 x\right)\cos x - 2\cos x \sin^2 x \\
&= \left(\cos^2 x - \left(1 - cos^2 x\right)\right)\cos x - 2\cos x\left(1 - cos^2 x\right) \\
&= 4\cos^3 x - 3\cos x
\end{aligned}$$

(b) For the specified intervals, solve the following equations. All your results should be written as fractions of π.

(i) $x \in [0 : 2\pi]$

$$\sin x = -\frac{1}{2}$$

$$RefA = \sin^{-1}(\frac{1}{2})$$

$$RefA = 30^\circ$$

3rd and 4th Quadrants :

$$x = 180 + 30 \quad x = 360 - 30$$

$$x = 210^\circ \quad x = 330^\circ$$

$$x = \frac{7\pi}{6} \quad x = \frac{11\pi}{6}$$

[5 marks]

(ii) $x \in [0 : \pi]$

$$\cos 4x = \frac{\sqrt{2}}{2}$$

$$RefA = \cos^{-1}\frac{\sqrt{2}}{2}$$

$$RefA = 45^\circ$$

1st and 4th Quadrants :

$$4x = 45^\circ \quad 4x = 315^\circ \quad 4x = 405^\circ \quad 4x = 675^\circ$$

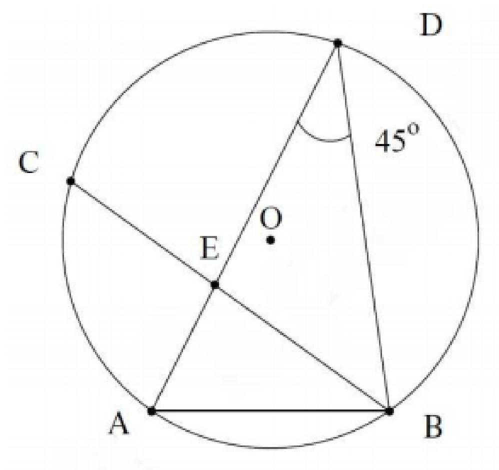
$$4x = \frac{\pi}{4} \quad 4x = \frac{7\pi}{4} \quad 4x = \frac{9\pi}{4} \quad 4x = \frac{15\pi}{4}$$

$$x = \frac{\pi}{16} \quad x = \frac{7\pi}{16} \quad x = \frac{9\pi}{16} \quad x = \frac{15\pi}{16}$$

[10 marks]

Question 6 (25 Marks)

Points A, B, C, and D are on a circle, see figure below, and < ADB = 45°. The lines (AD) and (BC) cross at point E. Show that if (AD) and (BC) are perpendicular and $|AB| = |AC|$, E is the centre of the circle. (Hint: compare the angles $< ACB$ and $< ADB$ and calculate other angles.)



$< ADB =< ACB = 45°$ and AB=AC
so $< BAC = 180- < ABC- < ACB = 90°$.
This means that [BC] is a diameter of the circle.
$< EBD = 180- < BED- < BDE = 45°$
so $< ABD =< ABE+ < EBD = 90$
so [AD] is a diameter of the circle.
E is at the intersection of two different diameters of the circle so E is the centre of the circle.

Section B	Contexts and Applications	150 Marks

Answer **both** questions from this section.

Question 7 (75 Marks)

A double soap bubble as shown in the picture can be modelled as shown in the picture on the next page. The objective of this question is to define the rules to plot a realistic soap bubble.

(a) Show that:

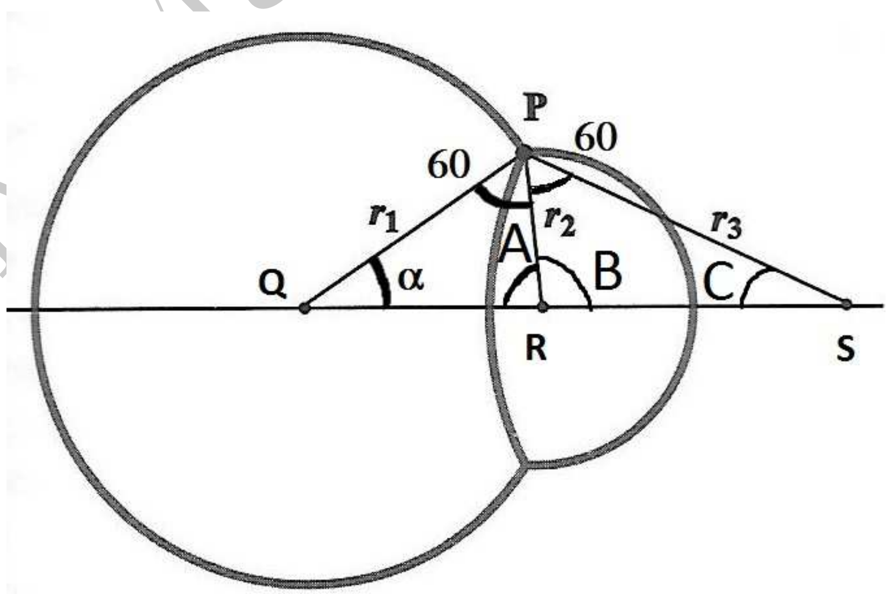
(i) $\sin(120-\alpha) = \sin(60+\alpha)$ [5 marks]

$$\sin(120-\alpha) = \sin[180-(120-\alpha)] = \sin(180-120+\alpha) = \sin(60+\alpha)$$

(ii) $\sin\alpha + \sin(60-\alpha) = sin(60+\alpha)$ [10 marks]

$$\begin{aligned}\sin\alpha + \sin(60-\alpha) &= 2\sin\left(\frac{\alpha+60-\alpha}{2}\right)\cos\left(\frac{\alpha-(60-\alpha)}{2}\right)\\ &= 2\sin(30)\cos(\alpha-30)\\ &= \cos(\alpha-30)\\ &= \sin(90-(\alpha-30))\\ &= \sin(120-\alpha)\\ &= \sin(60+\alpha)\end{aligned}$$

(b) In the figure below, express all the angles $<A$, $<B$ and $<C$ as a function of α [5 marks]



$<A = 120-\alpha$, $<B = 60+\alpha$, $<C = 60-\alpha$

(c) Show that

$$\frac{\sin(120-\alpha)}{r_1} = \frac{\sin\alpha}{r_2} \qquad \frac{\sin(60+\alpha)}{r_3} = \frac{\sin(60-\alpha)}{r_2}$$

hence show that

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2}$$

[15 marks]
Use the sin rule in triangle PQR

$$\frac{\sin(120-\alpha)}{r_1} = \frac{\sin\alpha}{r_2}$$

Use the sin rule in triangle PRS

$$\frac{\sin(60+\alpha)}{r_3} = \frac{\sin(60-\alpha)}{r_2}$$

Adding the two equations leads to

$$\begin{aligned}
\frac{\sin(120-\alpha)}{r_1} + \frac{\sin(60+\alpha)}{r_3} &= \frac{\sin\alpha}{r_2} + \frac{\sin(60-\alpha)}{r_2} \\
\Longrightarrow \frac{\sin(60+\alpha)}{r_1} + \frac{\sin(60+\alpha)}{r_3} &= \frac{\sin\alpha + \sin(60-\alpha)}{r_2} \\
\Longrightarrow \frac{\sin(60+\alpha)}{r_1} + \frac{\sin(60+\alpha)}{r_3} &= \frac{\sin(60+\alpha)}{r_2} \\
\Longrightarrow \frac{1}{r_1} + \frac{1}{r_3} &= \frac{1}{r_2}
\end{aligned}$$

(d) Show that if the segment of length r_2 on the graph is vertical, then $r_1 = r_3 = 2r_2$ [10 marks]
If the segment is vertical, $< PRQ =< PRS = 90°$ and $< PQR =< PSR = 30°$, so the two triangles PQR and PRS are similar. They have one common side so all corresponding sides are equal hence $r_1 = r_3$. Using the formula above,

$$\begin{aligned}
\frac{1}{r_1} + \frac{1}{r_3} &= \frac{1}{r_2} \\
\Longrightarrow \frac{1}{r_1} + \frac{1}{r_1} &= \frac{1}{r_2} \\
\Longrightarrow \frac{2}{r_1} &= \frac{1}{r_2} \\
\Longrightarrow r_1 &= 2r_2
\end{aligned}$$

(e) Calculate the common points to the two circles $\mathcal{C}_1$, centre $(-3\sqrt{3}, 0)$ and radius $R = 6$, and $\mathcal{C}_2$, centre $(3\sqrt{3}, 0)$ and radius $R = 6$. [10 marks]
$\mathcal{C}_1$, centre:$(-3\sqrt{3}, 0)$ and radius $R = 6$:

$$(x + 3\sqrt{3})^2 + (y - 0)^2 = 6^2$$
$$x^2 + 6\sqrt{3}x + 27 + y^2 = 36$$

$\mathcal{C}_2$, centre $(3\sqrt{3}, 0)$ and radius $R = 6$.

$$(x - 3\sqrt{3})^2 + (y - 0)^2 = 6^2$$
$$x^2 - 6\sqrt{3}x + 27 + y^2 = 36$$

To find the points of intersection we will first find the equation of the common chord.
To find the equation of the common chord just subtract the equations of the circles.

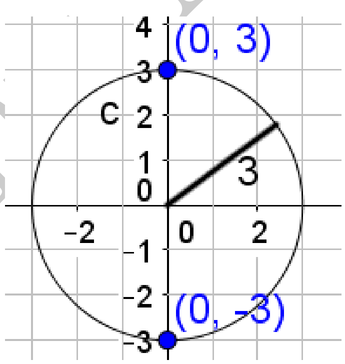
$$x^2 + 6\sqrt{3}x + 27 + y^2 = 36$$
$$-(x^2 - 6\sqrt{3}x + 27 + y^2 = 36)$$
$$-12\sqrt{3}x = 0$$
$$\text{Common Chord} : x = 0$$

Now find the point of intersection of the common chord and one of the circles, this will be the point of intersection of the two circles.

$$x = 0$$
$$x^2 + 6\sqrt{3}x + 27 + y^2 = 36$$
$$(0)^2 + 6\sqrt{3}(0) + 27 + y^2 = 36$$
$$27 + y^2 = 36$$
$$y^2 = 9$$
$$y = 3 \quad y = -3$$
$$x = 0 \quad x = 0$$

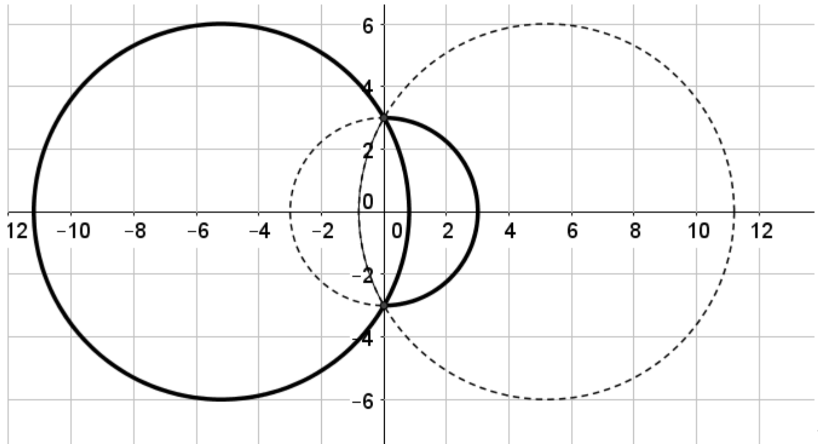
Points of intersection (0,3) and (0,-3)

(f) Calculate the equation of the circle which includes the two points (0,3) and (0,-3) and has a radius R=3. [10 marks]



$$x^2 + y^2 = 9$$

(g) Draw the three circles on the graph below. Do the three circles form a double soap bubble? Justify your answer with the results of the previous questions and outline a double soap bubble on your graph if there is one.



$r_1 = r_3 = 2r_2$ and top angle is $60°$. Two possible double soap bubbles. [10 marks]

Question 8 (75 Marks)

Following the 2008 crisis, banks now must evaluate how much they are likely to lose in worst case scenarios. To achieve this, banks study historical returns of classes of assets like interest rates or shares. The series below is an example of historical return series.

0.1117	0.2081	0.0386	-0.0094	-0.0551
-0.0806	0.2994	0.1807	-0.3281	-0.1595
-0.1811	0.7912	0.5629	0.1582	0.0023
0.4358	0.1658	-0.6029	-0.0745	0.2638
-0.0400	-0.2678	0.2752	-0.1410	-0.3154
1.123	-0.3504	-1.1048	-0.2688	-0.4734
0.4454	-0.1045	-0.2292	0.5919	0.1309
0.1764	-0.1537	0.8089	-0.7018	-0.0458
0.1040	-0.7324	0.5053	0.0886	-0.4094
-0.0619	0.1588	0.6462	-0.26964	-0.2645

(a) Describe the nature of the data. [5 marks]
Data numerical and continuous

(b) Using the information below, calculate the mean and standard deviation [5 marks]

$$\sum x_i = 0.846822 \quad \sum (x_i - \mu)^2 = 8.533262$$

where μ denotes the mean.

$$mean = \frac{\sum x_i}{50} \approx 0.016936$$

$$\sigma \approx \sqrt{\frac{\sum (x_i - \mu)^2}{50}} \approx 0.41312$$

(c) Complete the table below, then represent the distribution graphically [10 marks]

Interval	[-1.2;-1]	[-1;-0.8]	[-0.8;-0.6]	[-0.6;-0.4]	[-0.4;-0.2]	[-0.2;-0]
Frequences	1	0	3	2	8	12
Interval	[0;0.2]	[0.2;0.4]	[0.4;0.6]	[0.6;0.8]	[0.8;1]	[1;1.2]
Frequences	11	4	5	2	1	1

(d) Describe the distribution. Your description should include comments about the shape the mean, mode, standard deviation, tail behaviour and possibility to model the distribution with a normal distribution

[10 marks]
The returns vary between -1.1 and 1.1 approximately. The distribution is approximately symmetrical as shown by the average which is very close to zero. Most values concentrate around 0. The distribution then decreases to 0, more progressively on the positive side. The tail is slightly fatter on the positive side so the distribution is slightly right skewed. The mode occurs between -0.2 and 0. Since the data is nearly symmetrical and mor or less corresponds to a bell shape, it could be represented by a normal distribution accurately.

(e) Ideally, the distribution should be a normal distribution with a zero mean. If the series above was extended to 250 points, the corresponding mean and standard deviation are:

$$\mu = -0.0111 \qquad \sigma = 0.3885$$

(i) Calculate a 95% confidence interval for the value of the mean. Give your results with four significant digits
[10 marks]

$$\left[\mu - 1.96\frac{\sigma}{\sqrt{n}}\ ;\ \mu + 1.96\frac{\sigma}{\sqrt{n}}\right] = [-0.0593; 0.0371]$$

(ii) Test the claim that the mean is 0 at a 5% level of significance. Clearly specify both the null and alternative hypothesis and state your conclusion.
[10 marks]
H_0 mean is 0
H_1 mean is not 0
0 is inside the confidence interval, so accept H_0. Mean can be considered to be 0.

(f) For a normal distribution with the mean and standard deviation $\mu = -0.0111$ and $\sigma = 0.3885$, if a value is selected at random, find the probability that this value is

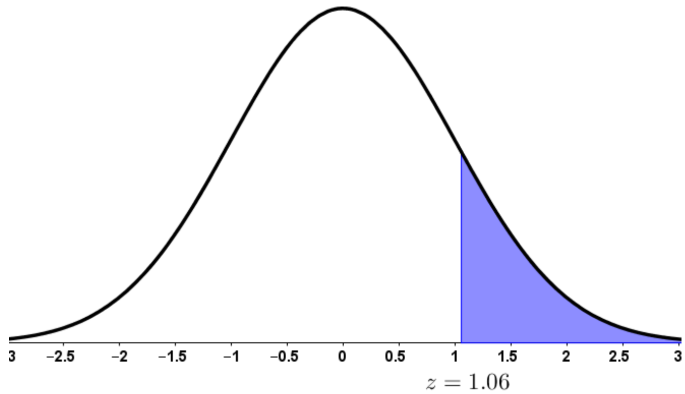
[15 marks].

(i) Above 0.4

$$P(X > 0.4)$$

$$z = \frac{0.4 - (-0.0111)}{0.3885}$$

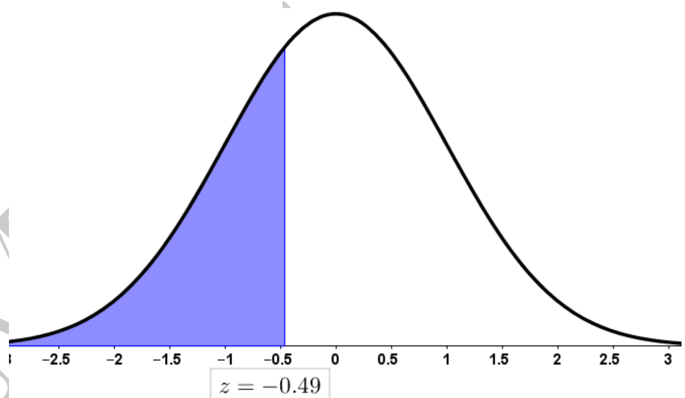
$$z = 1.06$$



$$P = 1 - 0.8554$$
$$\approx 0.145$$

(ii) Below -0.2

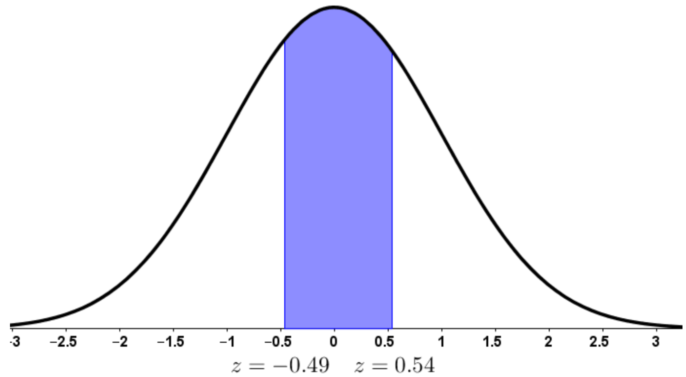
$$P(X < -0.2)$$
$$z = \frac{-0.2 - (-0.0111)}{0.3885}$$
$$z = -0.49$$



$$P = 1 - 0.6879$$
$$= 0.3121$$

(iii) between -0.2 and 0.2.

$$P(-0.2 < X < 0.2)$$
$$X = -0.2 \quad z = \frac{-0.2 - (-0.111)}{0.3885}$$
$$z = -0.49$$
$$X = 0.2 \quad z = \frac{0.2 - (-0.111)}{0.3885}$$
$$z = 0.54$$



$$P = .0754 - (1 - .6879)$$
$$P = 0.3933$$

(iv) If you have a set of 50 values randomly chosen (individually) from a normal distribution with mean and standard deviation $\mu = -0.0111$ and $\sigma = 0.3885$, how many should be above 0.4, below -0.2 and between -0.2 and 0.2? Compare with the results of part (c). Does this mean the distribution is normal?

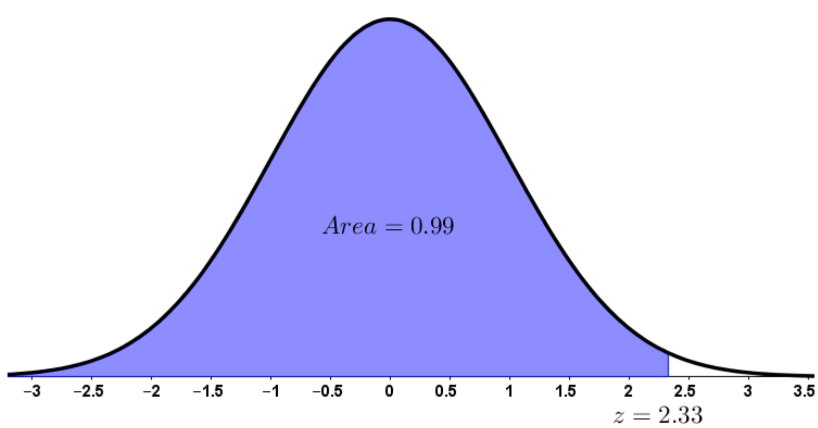
$$\text{Above } 0.4 = 0.145 \times 50 \approx 7$$
$$\text{Below -0.2} = 0.3121 \times 50 \approx 16$$
$$\text{Between -0.2 and } 0.2 = 0.3933 \times 50 \approx 20$$

According to the analysis in part (c), there are 14 values below -0.2, 4 values above 0.4 and 23 values between -0.2 and 0.2. These results are quite close to the theoretical values. Given that there are only 50 points in the series, this confirms that the series could be represented by a normal distribution.

(g) Following your study, a bank represents the returns of a share with a normal distribution with a mean $\mu = 0$ and a standard deviation $\sigma = 0.4$. Calculate the value R_{99} verifying

$$P(X < R_{99}) = 0.99$$

and explain what this value represents. [5 marks]



$$z = 2.33$$
$$2.33 = \frac{x - 0}{0.4}$$
$$R_{99} = 0.932$$

99% of returns will be below 0.932

(h) The value at risk is defined as

$$VaR = \left| Value \times \frac{R_{99}}{100} \right|$$

The bank has a €1,000,000 investment with returns following a normal distribution with the mean $\mu = 0$ and standard deviation $\sigma = 0.4$. Calculate the corresponding value at risk. What could make this value higher? [5 marks]

$$VaR = €9320$$

This value would get higher if R_{99} increased, this means if the returns had a higher volatility