Leaving Certificate Examination, 2014

Sample paper prepared by Leamy Maths Community

# Mathematics

# Project Maths - Phase 2

Paper 1

# Solutions

Higher Level

Saturday 25 April

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Leamy Maths Community

300 marks

| Section A | Concepts and Skills | 100 Marks |
|---|---|---|

Answer **all six** questions from this section.

**Question 1** **(25 Marks)**

**(a)** If (x+1) is a factor in the cubic function $f : x \mapsto x^3 - 2x^2 + ax + 2$, find the value of a.

$$f(-1) = -1 - 2 - a + 2 = 0 \Longrightarrow a = -1$$

5 marks

**(b)** Find the other two roots of the function.

$$\begin{aligned} f(x) &= (x+1)\left(x^2 - 3x + 2\right) \\ f(x) = 0 &\Longrightarrow x + 1 = 0 \quad x^2 - 3x + 2 = 0 \\ & \quad\; x = -1 \quad x = 2 \quad x = 1 \end{aligned}$$

5 marks + 5 marks

**(c)** Using parts (a) and (b) or otherwise, solve the equation

$$y^6 - 2y^4 + ay^2 + 2 = 0$$

where $a$ is the value calculated above(Hint substitute $x = y^2$)

$$\begin{aligned} &y^2 = 0 \quad y^2 = 2 \quad y^2 = -1 \text{ (impossible)} \\ &y = 0 \quad y = \sqrt{2} \quad y = -\sqrt{2} \end{aligned}$$

5 marks + 5 marks

**Question 2** **(25 Marks)**

**(a)** Expand $(\cos\theta + i \sin\theta)^2$.

$$\begin{aligned} (\cos\theta + i\sin\theta)^2 &= \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta \\ &= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \end{aligned}$$

5 marks

**(b)** Using de Moivre's theorem, calculate $(\cos\theta + i\sin\theta)^2$ and show that

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\cos\theta\sin\theta \end{aligned}$$

$$
\begin{aligned}
z = \cos\theta + i\sin\theta \Longrightarrow z^2 &= \cos 2\theta + i\sin 2\theta \\
&= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta
\end{aligned}
$$

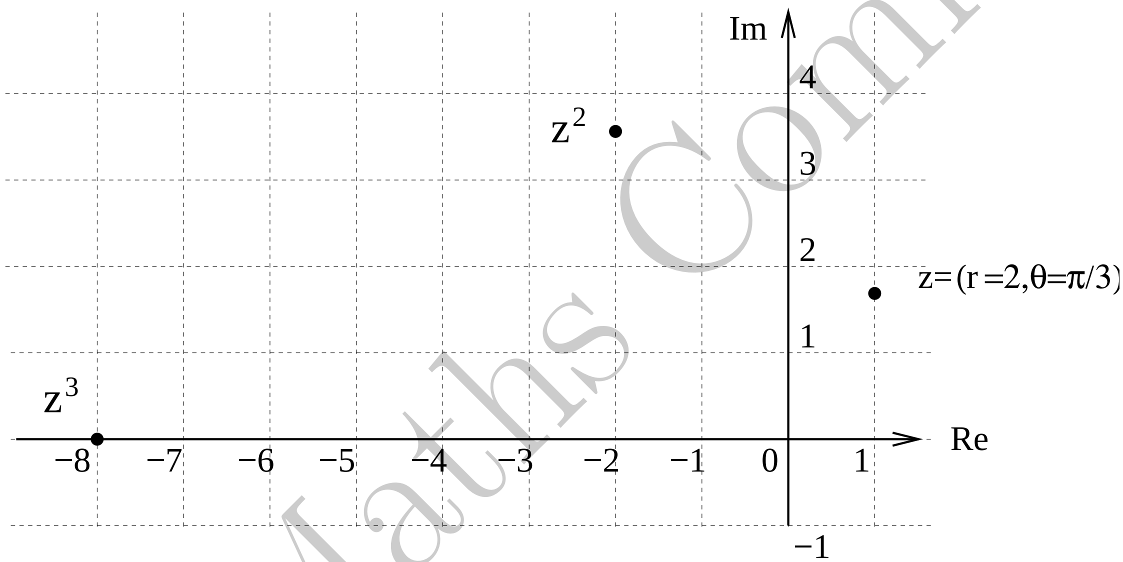
5 marks

Comparing real and imaginary parts leads to

$$
\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
\sin(2\theta) &= 2\cos\theta\sin\theta
\end{aligned}
$$

5 marks

**(c)** The complex number $z$ is defined by its modulus $|z| = 2$ and its argument $Arg(z) = \pi/3$. Using de Moivre's theorem, place the points $z^2$ and $z^3$ on the graph. Label the points and give their coordinates in polar form




5 marks

$$
\begin{aligned}
z^2 &= 4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -2 + 2\sqrt{3}i \\
z^3 &= 8\left(\cos(\pi) + i\sin(\pi)\right) = -8i
\end{aligned}
$$

5 marks

**Question 3** **(25 Marks)**

**(a)** Solve the following inequalities

**(i)**

$$
\frac{x-2}{x-3} < 2\ .
$$

$$\frac{x-2}{x-3} < 2 \implies x^2 - 7x + 12 > 0$$
$$\implies x < 3 \qquad z > 4$$

10 marks

**(ii)**

$$e^{2x+1} > 4 \ .$$

$$x > \frac{\ln 4 - 1}{2} \approx 0.19315$$

5 marks

**(b)** Solve the simultaneous equations

$$\begin{cases} x + y + z = 4 \\ x + 2y + 3z = 9 \\ 2x + 4y - z = -3 \end{cases}$$

$$x = 2 \quad y = -1 \quad z = 3$$

10 marks

**Question 4** **(25 Marks)**

**(a)** The population of the urban area in Limerick grows exponentially

$$p = Ae^{Bt}$$

where $p$ is the population and $t$ is the time in years. In 2011, there were 91,000 people in the urban area of Limerick. Assuming there are 92,500 in 2014 (3 years later), calculate A and B.

$$A = 91,000 \quad B = \frac{\ln 92500 - \ln 91000}{3} \approx 0.00545$$

$$p = 91000e^{0.00545t}$$

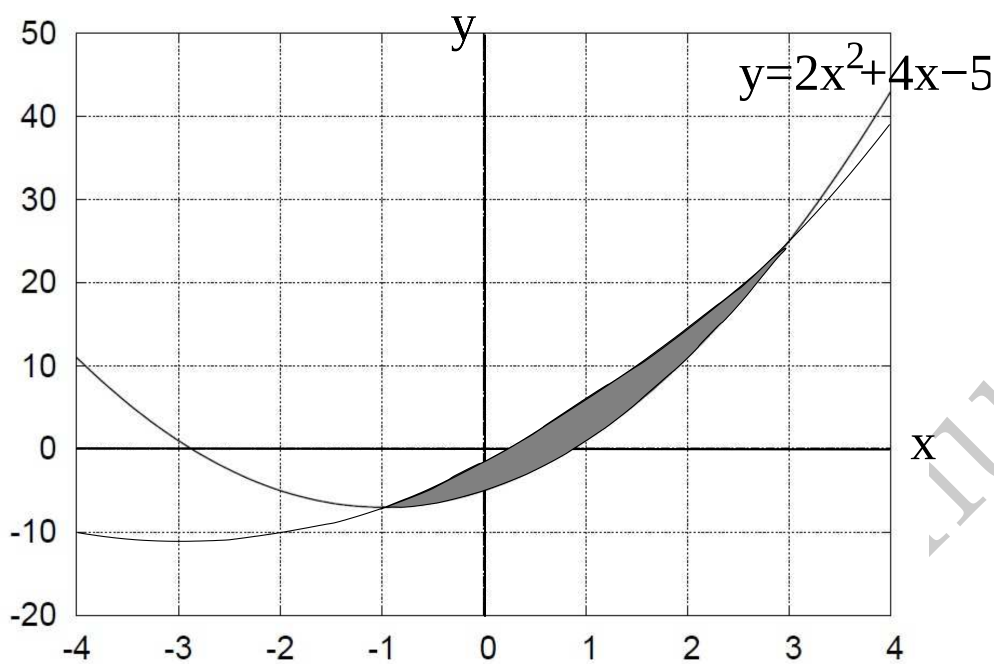
15 marks

**(b)** Using the values of A and B calculated above, when will the population of the urban area reach 100,000 people?

$$100000 = 91000e^{0.00545t} \implies t = \frac{\ln 100000 - \ln 91000}{0.00545} \approx 17.3$$

This will happen during 2028

10 marks




**Question 5** **(25 Marks)**

**(a)** Find graphically for what values of $x$ do the two curves $y = 2x^2 + 4x - 5$ and $y = x^2 + 6x - 2$ cross and verify your answers algebraically.
5 marks

$$2x^2 + 4x - 5 = x^2 + 6x - 2 \Longrightarrow x^2 - 2x - 3 = 0 \Longrightarrow x = 3 \quad x = -1$$

5 marks

**(b)** Shade the area between the two curves and use calculus to evaluate the value of the area.

$$\begin{aligned} A &= \int_{-1}^{2} \left[ (x^2 + 6x - 2) - (2x^2 + 4x - 5) \right] dx \\ &= \int_{-1}^{2} (-x^2 + 2x + 3) dx) \\ &= \left[ -\frac{x^3}{3} + x^2 + 3x \right]_{-1}^{3} \\ &= \frac{32}{3} \end{aligned}$$

10 marks + 5 marks

**Question 6** **(25 Marks)**

**(a)** A dairy farmer has x cows. Each cow produces an income of 20 euros per day. It costs the farmer $x^2 - 120x + 2100$ euros everyday to house and feed the cows. Show that the daily profit is

$$\pi(x) = -x^2 + 140x - 2100$$

Justify your calculation.

$$\begin{aligned} Profit &= Income - Costs \\ &= 20x - (x^2 - 120x + 2100) \\ &= -x^2 + 140x - 2100 \end{aligned}$$

5 marks

**(b)** What number of cows maximises this profit?

$$\begin{aligned} \pi' &= -2x + 140 \\ \pi' &= 0 \Longrightarrow x = 70 \end{aligned}$$

5 marks + 5 marks

**(c)** Calculate the maximum profit and the break even points.

$$\begin{aligned} \pi(70) &= 2800 \\ \pi &= 0 \Longrightarrow x = \frac{-140 \pm \sqrt{140^2 - 4(-1)(-2100)}}{-2} \Longrightarrow x \approx 17 \quad x \approx 123 \end{aligned}$$

3 marks +7 marks

| Section B | Contexts and Applications | 150 Marks |
|---|---|---|

Answer **all three** questions from this section.

**Question 7** **(50 Marks)**

Séan takes out a three year loan at the bank to buy his new €20000 car.

**(a)** Every month, Séan pays an interest rate of 0.5%

**(i)** Calculate the annual equivalent interest rate

$$(1+r)^{12} = 1+i \Longrightarrow i = 1.005^{12} - 1 = 6.17\%$$

5 marks

**(ii)** Another bank offers Séan an annual equivalent rate of 6.5%. What should he do?
Keep the 6.17% offer

5 marks

**(b)** Séan will make his first payment of value $A$ in a month time. Using the monthly interest rate r=0.5%

**(i)** Calculate the present value of the first payment (payment he will make in one month)

$$P = \frac{A}{1.005}$$

5 marks

**(ii)** Using **induction**, show that the sum of the present values of monthly payments is

$$PV = A\frac{(1+r)^n - 1}{r(1+r)^n}$$

Calculate the value for 3 years (=36 months)

- The property is true for n=1 as

$$A\frac{(1+r)^1 - 1}{r(1+r)^1} = A\frac{r}{r(1+r)} = \frac{A}{1+r}$$

- Assume true for n

$$\frac{A}{1+r} + \frac{A}{(1+r)^2} + \ldots + \frac{A}{(1+r)^n} = A\frac{(1+r)^n - 1}{r(1+r)^n}$$

- Prove it is true at level n+1

$$\begin{aligned}\frac{A}{1+r}+\frac{A}{(1+r)^2}+\ldots+\frac{A}{(1+r)^n}+\frac{A}{(1+r)^{n+1}} &= A\frac{(1+r)^n-1}{r(1+r)^n}+\frac{A}{(1+r)^{n+1}}\\ &= \frac{A}{(1+r)^n}\left[\frac{(1+r)^n-1}{r}+\frac{1}{1+r}\right]\\ &= A\frac{(1+r)^{n+1}-1}{r(1+r)^{n+1}}\end{aligned}$$

  This is the property at level n+1.
- The property is inductive, true for n=1, so it is true for $n \geq 1$

- Value for 3 years: PV=

$$PV = A\frac{1.005^{36}-1}{0.005(1.005)^{36}} \Longrightarrow PV = 32.871A$$

15 marks

**(c)** What monthly payment should Séan pay back every month?

$$20000 = A\frac{1.005^{36}-1}{0.005(1.005)^{36}} \Longrightarrow A = 608.44$$

8 marks

**(d)** How much will he pay back in total?

$$608.44 \times 36 = €21,903.34$$

7 marks

**(e)** Why are the two figures different?
Interest rates
5 marks

**Question 8** **(50 Marks)**

A producer of hot water cylinders wants to build his product with a brand new material with excellent thermal properties but quite expensive. His objective is to manufacture cylindrical hot water cylinders of radius $r$ and height $h$.

**(a)** To start with, the producer wants to know the optimal shape.

**(i)** Draw a picture of the cylinder and calculate its volume $V$ and surface $A$ (including the top and bottom surfaces) as a function of the radius $r$ and the height $h$.

$$V = \pi r^2 h \qquad A = 2\pi r h + 2\pi r^2$$

5 marks

**(ii)** Show that the volume may be expressed as

$$V = \frac{r}{2}\left(A - 2\pi r^2\right)$$

$$A = 2\pi r h + 2\pi r^2 \Longrightarrow h = \frac{A - 2\pi r^2}{2\pi r} \Longrightarrow V = \pi r^2 h = \pi r^2 \frac{A - 2\pi r^2}{2\pi r} = \frac{r}{2}\left(A - 2\pi r^2\right)$$

5 marks

**(iii)** Calculate the value of the radius which maximises the volume of the cylinder as a function of $A$

$$V = \frac{Ar}{2} - \pi r^3 \Longrightarrow V' = \frac{A}{2} - 3\pi r^2 \qquad V' = 0 \Longrightarrow r^2 = \frac{A}{6\pi} \Longrightarrow r = \sqrt{\frac{A}{6\pi}}$$

5 marks

**(iv)** Calculate the value of the corresponding values for the height.

$$h = \sqrt{\frac{2A}{3\pi}}$$

5 marks

**(v)** Calculate the corresponding value of the volume.

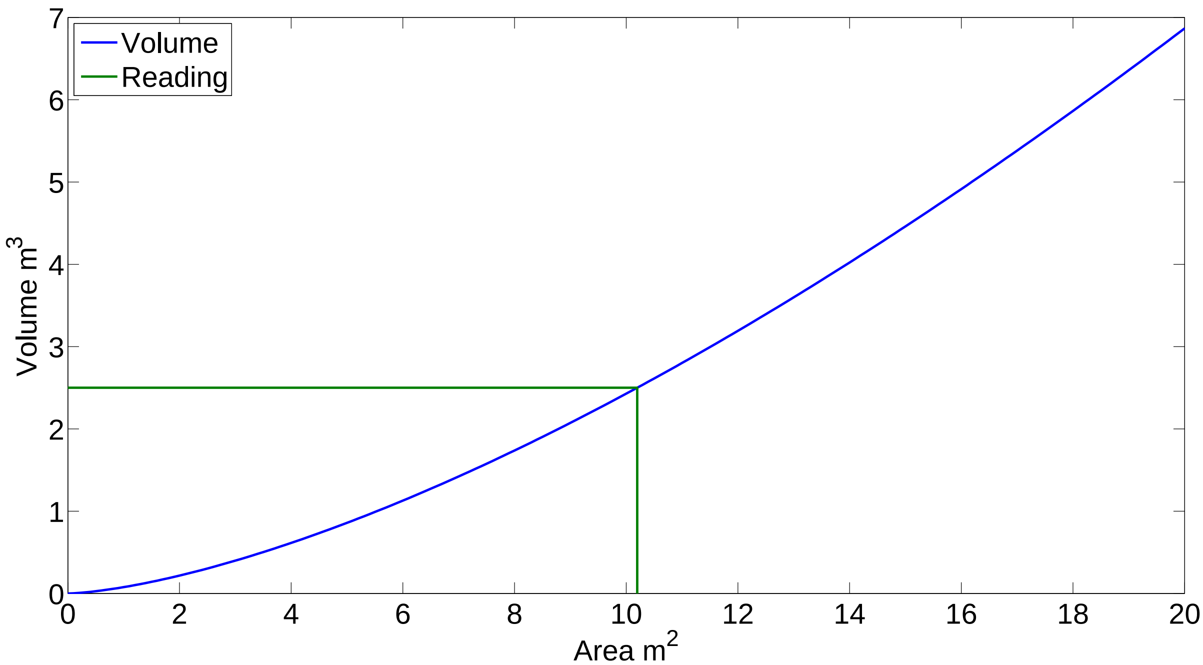
$$V = \frac{A^{3/2}}{3\sqrt{6\pi}}$$

5 marks

**(b)** The producer now wants to optimise its offer and determine the optimal area he should use.

**(i)** Plot the volume of the tank as a function of areas between 0 and 20m$^2$ using appropriate scales

$$V = \frac{A^{3/2}}{3\sqrt{6\pi}}$$




15 marks

**(ii)** Read on the curve what area corresponds to a volume of 2.5m$^3$. Explain your method
Read just over 10, around 10.1
5 marks

**(iii)** Retrieve the result by solving the appropriate equation.

$$2.5 = \frac{A^{3/2}}{3\sqrt{6\pi}} \Longrightarrow A = 7.5^{2/3}\sqrt[3]{6\pi} \approx 10.197$$

5 marks

**Question 9** **(50 Marks)**

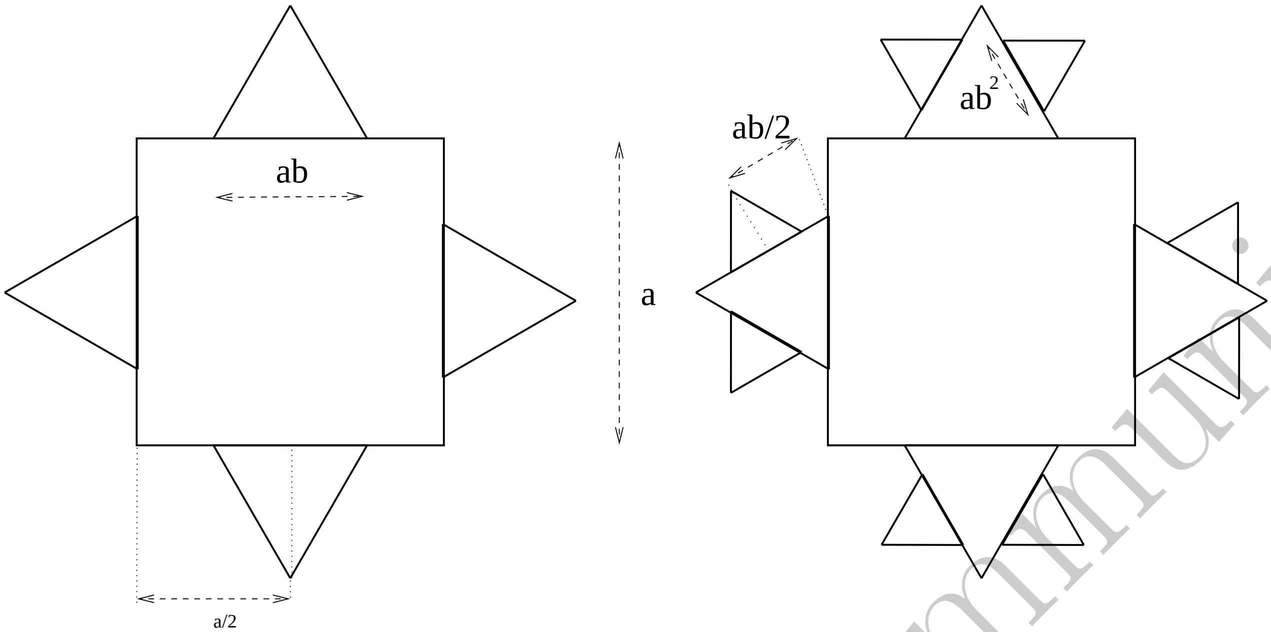
**(a)** Show that the area of an equilateral triangle with sides of dimension a is

$$Area = \frac{a^2\sqrt{3}}{4}$$

$$A = \frac{1}{2}ah \qquad h = a\sin 60 \Longrightarrow A = \frac{1}{2}a^2\sin 60 = \frac{a^2\sqrt{3}}{4}$$

5 marks

**(b)** A geometric figure is constructed as shown below




Level 1 Level 2

**(i)** Explain how you would construct the next level
Add 16 equilateral triangles of side length $ab^3$
5 marks

**(ii)** Complete the table below

| | | | | | |
|---|---|---|---|---|---|
| Level | 1 | 2 | 3 | 4 | 5 |
| Number of added triangles | 4 | 8 | 16 | 32 | 64 |
| Measure of the side of an added triangle | $ab$ | $ab^2$ | $ab^3$ | $ab^4$ | $ab^5$ |

8 marks

**(iii)** Show that the side of a triangle added at level $n$ is $ab^n$. Calculate the area of an added triangle at this level.

- n=1, $l = ab$
- n=2, $l = ab^2$
- n=3, $l = ab^3$

This is a geometric series with first term ab and r=b

$$T_n = abr^{n-1} = abb^{n-1} = ab^n$$

7 marks

**(iv)** Show that all the triangles added at level $n$ have a total area of

$$Area_n = \frac{2^{n+1}\sqrt{3}a^2 \times b^{2n}}{4}$$

The number of triangles added is a geometric series

$$T_n = 2^{n+1}$$

The triangles at level n are of size $ab^n$, the total area added is therefore

$$Area_n = 2^{n+1} \times \frac{(ab^n)^2\sqrt{3}}{4} = \frac{2^{n+1}\sqrt{3}a^2 \times b^{2n}}{4}$$

8 marks

**(v)** Show that the series $Area_n$ is a geometric series of ratio $2b^2$

$$Area_n = \frac{2^{n+1}\sqrt{3}a^2 \times b^{2n}}{4} \qquad Area_{n+1} = \frac{2^{n+2}\sqrt{3}a^2 \times b^{2n+2}}{4} \Longrightarrow \frac{Area_{n+1}}{Area_n} = 2b^2$$

7 marks

**(vi)** For what values of $b$ does the sum of $Area_n$ to infinity exist? Give the the value of this infinite sum

The sum exists for $|r| < 1$ meaning

$$|2b^2| < 1 \Longrightarrow 2b^2 < 1 \Longrightarrow b^2 < \frac{1}{2} \Longrightarrow -\frac{1}{\sqrt{2}} < b < \frac{1}{\sqrt{2}}$$

Since $b > 0$, the possible values for b verify

$$0 < b < \frac{1}{\sqrt{2}}$$

10 marks