Leaving Certificate Examination, 2016

Sample paper prepared by Leamy Maths Community

Mathematics

Paper 2

Higher Level

Sunday 1 May

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Leamy Maths Community

Solutions

300 marks

Sample Instructions

There are two sections in this examination paper:

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	2 questions

Answer questions as follows:

In Section A, answer all six questions.
In Section B, answer all two questions.

Write your answers in the spaces provided in this booklet. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the booklet of *Formulae and Tables*. You must return it at the end of the examination.

You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

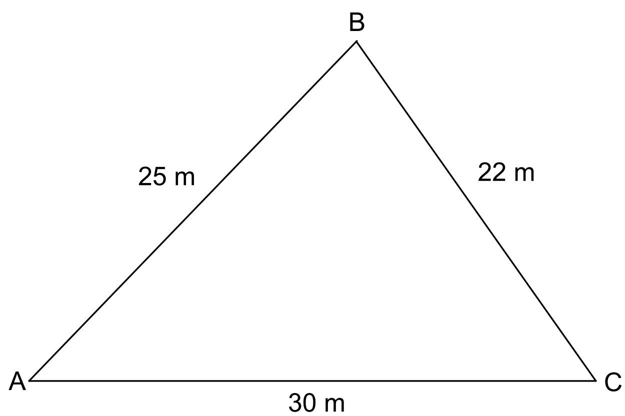
Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	150 Marks

Answer **all six** questions from this section.

Question 1 (25 Marks)

(a) A promoter wants to by a plot of land delimited by 3 road which form the triangle shown in the picture above where $|AB| = 25\text{m}$, $|BC| = 22\text{m}$ and $|AC| = 30\text{m}$,



(i) Calculate the value in degrees of the angle $< CAB$ correct to two decimal places

$$BC^2 = AB^2 + AC^2 - 2AB \times AC \cos(< CAB)$$
$$\implies \cos(< CAB) = \frac{25^2 + 30^2 - 22^2}{2 \times 25 \times 30} = 0.694 \implies < CAB = 46.05°$$

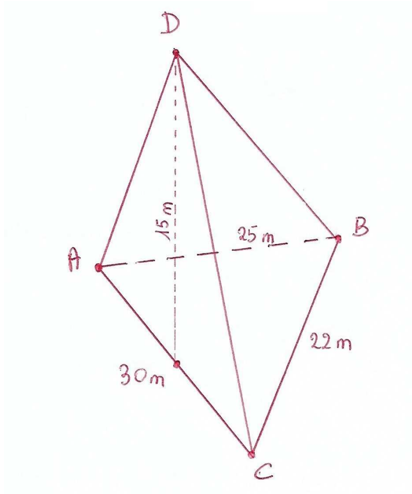
Marks: 0, 4, 7, 10

(ii) Hence or otherwise calculate the area of the plot correct to one decimal place.

$$S = \frac{1}{2} 25 \times 30 \times \sin < CAB = 270m^2$$

Marks: 0, 2, 3, 5

(b) The promoter wants to build a pyramid shaped building with a triangular base. The highest point D is located 15m vertically above the middle of $[AC]$.



(i) Calculate the volume of the pyramid

$$V = \frac{1}{3}Sh = \frac{1}{3}270 \times 15 = 1330m^3$$

Marks: 0, 2, 3, 5

(ii) The triangle ACD is to be covered with a special paint costing €100 per square meter. How much will it cost to complete the painting?

$$Cost = \frac{1}{2}30 \times 15 \times 100 = €22,500$$

Marks: 0, 2, 3, 5

Question 2 (25 Marks)

Line 1	y=2x+3
Line 2	2x+y-7=0
Line 3	x+2y-1=0
Line 4	4y=3x+5
Line 5	$\frac{1}{3}$y=$\frac{1}{4}$x+2
Line 6	y=x+5

(a) Which lines are parallel (justify your answer)?
Lines 4 and 5 are parallel as they both have the same slope: 3/4
Marks: 0, 2, 3, 5

(b) Which lines are perpendicular (justify your answer)?
Lines 1 and 3 are perpendicular as

$$m_1 \times m_3 = 2 \times \left(-\frac{1}{2}\right) = -1$$

Marks: 0, 2, 3, 5

(c) Calculate the distance between Line 4 and point $(10, -5)$.

$$d = \frac{|3(10) - 4(-5) + 5|}{\sqrt{3^2 + 4^2}} = \frac{55}{5} = 11$$

Marks: 0, 2, 3, 5

(d) What point is common to Line 1 and Line 3? Calculate the result analytically and then verify your answer graphically.

x=-1, y=1
Marks: 0, 2, 3, 5

(e) Calculate the equation of the lines which make a 45° angle with Line 6 and has the same y intercept as Line (5).
Intercept (0,6)
Slope verifies

$$\tan 45 = \pm\frac{m-1}{m+1} \Longrightarrow m = 0 \qquad or \qquad m = \infty$$

Two lines: $y = 6$ and $x = 0$
Marks: 0, 2, 3, 4, 5

Question 3 (25 Marks)

In a fair, a new game has been introduced. You throw two dice and add the two values:

- If the total is 5 or below, you win €5,
- If the two dice have the same value and the total of the two dice is above 5, you win €10,
- You don't win anything otherwise.

(a) Probabilities

(i) Calculate the probability to win €5.

$$p = \frac{10}{36} = \frac{5}{18}$$

Marks: 0, 2, 3, 5

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(ii) Calculate the probability to win €10.

$$p = \frac{4}{36} = \frac{1}{9}$$

Marks: 0, 2, 3, 5

(iii) Using the results above, calculate the probability to win nothing.

$$p = 1 - \frac{5}{18} - \frac{1}{9} = \frac{11}{18}$$

Marks: 0, 2, 3, 5

(b) Price of the ticket

(i) On average, how much will a player win when playing this game?

$$E = \frac{10}{36} \times 5 + \frac{4}{36} \times 10 = \frac{90}{36} = €2.5$$

Marks: 0, 2, 3, 5

(ii) The organisers consider charging €2 to play this game. Does this make sense? Justify you answer.
No on average they would lose €0.5 per game.
Marks: 0, 2, 3, 5

(iii) They decide to charge €5 to play the game. If during the fair, 100 people play this game, what will be the expected benefit for this game? Justify your answer.

$$Benefit = 100 \times (5 - 2.5) = €250$$

Marks: 0, 2, 3, 5

Question 4 (25 Marks)

(a) Equation of a circle

(i) Find the centre and radius of the circle $\mathcal{C}$

$$x^2 - 6x + y^2 - 8y = 0$$

Centre (3,4), Radius=5

(ii) Show that the point $(-1,7)$ is on circle $\mathcal{C}$

$$(-1)^2 - 6(-1) + (7)^2 - 8(7) = 0$$

The point is on the circle
Marks: 0, 2, 5

(iii) Calculate the equation of the tangent to circle $\mathcal{C}$ at point $(-1,7)$

$$4x - 3y + 25 = 0$$

Marks: 0, 3, 5, 7

(b) Touching circles

(i) Calculate the equation of the circle radius 10 externally touching circle $\mathcal{C}$ at point $(-1,7)$.
Distance 5 corresponds to -4 in the x direction and +3 in the y direction (to go from the centre to point (-1,7)). Do this twice to touch externally and once in the other direction for circles touching internally. Here, centre is (-9, 13). Equation is

$$(x+9)^2 + (y-13)^2 = 10^2$$

Marks: 0, 2, 4, 6, 8

(ii) Calculate the equation of the circle radius 10 internally touching circle $\mathcal{C}$ at point $(-1,7)$.
Here centre is (7,1). The equation of the circle is

$$(x-7)^2 + (y-1)^2 = 0^2$$

Marks: 0, 2, 4, 6, 8

Question 5 (25 Marks)

A survey among 100 randomly chosen people shows that they spent on average 84 minutes in their car per day with a standard deviation for this sample of 25 minutes.

(a) The people carrying the survey had two options: they could either randomly phone people or ask people randomly at a toll plaza. Which method would you advise?
Use the phone. People at the plaza are already driving. The survey would be biased.
Marks: 0, 2, 5

(b) Construct a 95% confidence interval for the average amount of time spent by motorists in their car.

$$\left[84 - \frac{25}{\sqrt{10}}, 84 + \frac{25}{\sqrt{10}}\right] = [79.1, 88.9]$$

Marks: 0, 4, 7, 10

(c) An advertiser wants to say that people spent 90 minutes per day in their car. Based on the survey, test the advertiser's claim. Clearly state both the null and alternative hypothesis and state your conclusion.
H_0 people spend 90 minutes in their car
H_1 people do not spend 90 minutes in their car
90 is outside the confidence interval, so reject H_0.
Marks: 0, 4, 7, 10

Question 6 (25 Marks)

(a) Construct the circumcircle of the triangle below.

Marks: 0, 4, 7, 10

(b) Using the triangle ABC below, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$Area = ab\sin C = bc\sin A = ac\sin B$$

and divide by abc. This leads to the desired result.
Marks: 0, 4, 7, 10, 15

Section B	Contexts and Applications	150 Marks

Answer **both** questions from this section.

Question 7 (75 Marks)

Until August 2015, the Irish lottery used 45 balls for each draw. Two extra balls, were added in September 2015, making the total number of balls 47. Balls are numbered 1 to 45 or 1 to 47. In each draw, 6 balls are chosen randomly among the 45 or 47 balls available. You win the jackpot if you choose the 6 correct numbers. The order in which the balls appear during a draw is not important.

(a) Choosing 6 correct numbers

(i) How many ways are there to select 6 balls from 45?

$$N = \binom{45}{6} = 8145060$$

Marks: 0, 2, 3, 5

(ii) How many ways are there to select 6 balls from 47? Calculate the % increase in number of possible combinations when the 2 extra balls were added to the draw.

$$N = \binom{47}{6} = 10737573 \quad \Longrightarrow \quad Increase = 100 * \left(\frac{10737573}{8145060} - 1\right) = 31.8\%$$

Marks: 0, 2, 3, 5

(b) In order to win a prize, you need to pick 3, 4, 5 or 6 correct numbers from a draw of 6 numbers (or balls).

(i) Show that there are 182780 number of ways to have 3 correct numbers and 3 incorrect balls, from a total of 45 balls. Hence calculate the probability of selecting 3 correct and 3 incorrect balls, from a total of 45.

$$N = \binom{6}{3}\binom{39}{3} = 182780 \Longrightarrow p = \frac{182780}{8145060} = 0.02244$$

Marks: 0, 2, 3, 5

(ii) Calculate the probability of selecting 3 correct numbers (and therefore 3 incorrect) from 47 balls.

$$N = \binom{6}{3}\binom{41}{3} = 10660 \times 20 = 213200 \Longrightarrow p = \frac{213200}{10737573} = 0.01986$$

Marks: 0, 2, 3, 5

(c) For 47 balls the following and prizes have been calculated:

Nb of correct numbers	Probability	Prize
< 3	0.979	0
3	0.01986	€10
4	0.00115	€200
5	2.29×10^{-5}	€20,000
6	$9.31\ \times10^{-8}$	€2,000,000

(i) Calculate the expected gain and find the minimum price to play a 6 ball combination.

$$\begin{aligned} E &= 0.01986 \times 10 + 0.00115 \times 200 + 2.29 \times 10^{-5} \times 20,000 + 9.31 \times 10^{-8} \times e2,000,000 \\ &= 0.1986 + .023 + 0.458 + 0.1862 = 1.0728 \end{aligned}$$

The minimum price is €1.0728.
Marks: 0, 4, 7, 10

(ii) 1,000,000 sets of six numbers were played for a given draw. Entering the game with a set of 6 numbers costs €2. On average, how much money can the lottery give to good causes, after prize money has been given to winners.

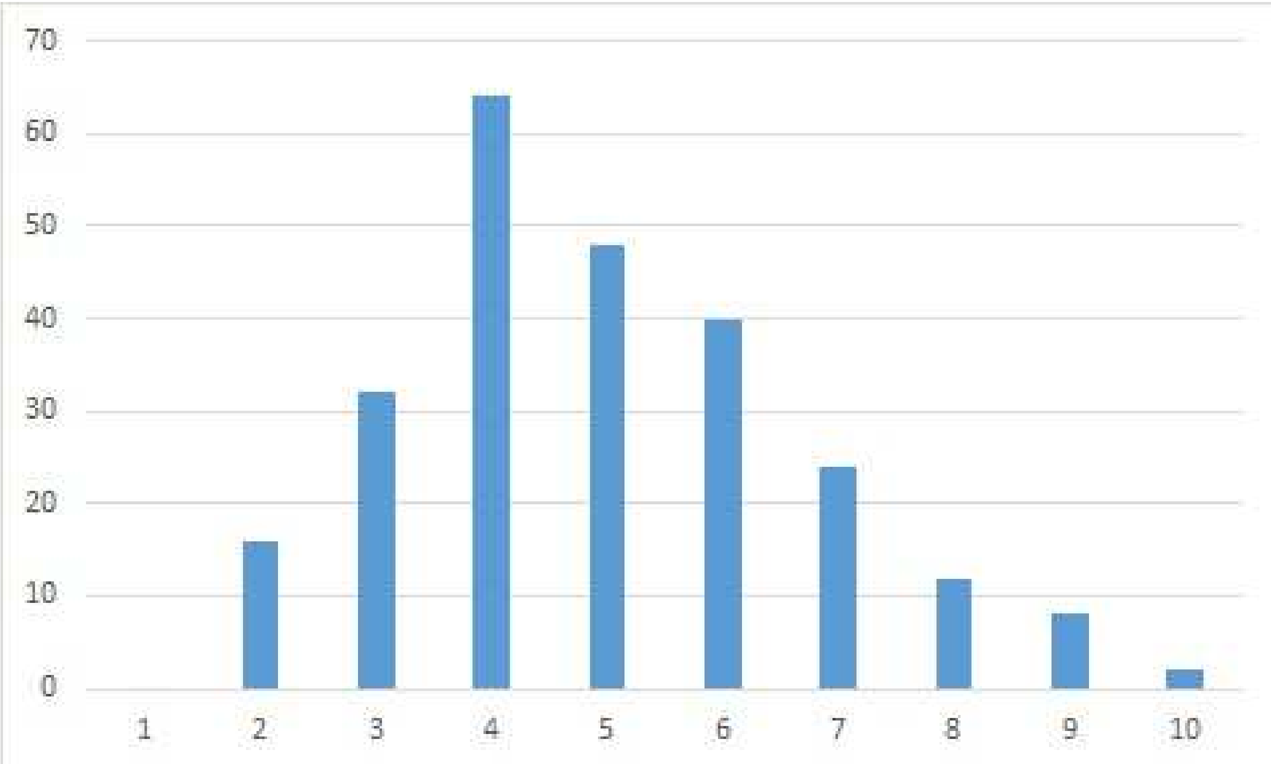
$$E = 1000000 \times (2 - 1.0728) = €927200$$

Marks: 0, 2, 3, 5

(d) Players can play between 2 and 10 sets of 6 numbers for a draw. Below is the distribution of the number of sets played by people for a given draw.

Nb of sets played	2	3	4	5	6	7	8	9	10
Nb of players ($\times1,000$)	16	32	64	48	40	24	12	8	2

(i) Create a suitable graphical representation for the number of grid played.
Marks: 0, 4, 7, 10



(ii) Describe the distribution. This should include information about such as shape, range, estimation for the median value. What is the position of the mean relative to the median. You should also include the value of the mean and standard deviation and discuss if the data can be represented with a normal distribution.
The number of grids played ranges from 2 to 10. The distribution is right skewed and rapidly increases for n=2 to n=4 and then progressively decreases till n=10 with a long right tail. The data is clearly not symmetrical and therefore it can not be represented by a normal distribution accurately. The most common choice is 4 grids with 64,000, 4 is therefore the mode. The median is 5 while the mean can be calculated as

$$m = \frac{16(2) + 32(3) + 64(4) + 48(5) + 40(6) + 24(7) + 12(8) + 8(9) + 2(10)}{16 + 32 + 64 + 48 + 40 + 24 + 12 + 8 + 2} = 4.96$$

The standard deviation can be calculated as approximately 1.76.
Marks: 0, 5, 7, 10, 15

(e) Jane has her lucky numbers. She noticed that she wins 20% of the time. In a year, she plays 10 times.

(i) Can Jane's luck be considered a Bernoulli trial? Specify the conditions for Bernoulli trials (also called binomial trials) and then conclude if it is possible to use this method here.
Possible because

- Two outcomes

- Fixed number of trials
- independent events
- constant probabilities

Marks: 0, 2, 3, 5

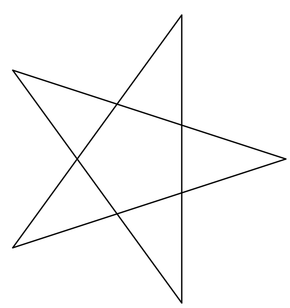
(ii) What is the probability that she wins three times or more?

$$\begin{aligned} P &= 1 - \left[\binom{10}{0} 0.8^{10} 0.2^{0} + \binom{10}{1} 0.8^{9} 0.2^{1} + \binom{10}{2} 0.8^{8} 0.2^{2} \right] \\ &= 1 - (0.1074 + 0.2684 + 0.3020) = 0.3222 \end{aligned}$$

Marks: 0, 4, 7, 10

Question 8 (75 Marks)

Christmas is far away but you can still prepare early for it. Here is how you can draw a five branched star. A five branched star has 5 equally spaced branches which can be easily be drawn with a compass and a ruler. As there are five equally spaced branched in the star, the angle $2\pi/5$ plays a key role. In this question, **you should not use your calculator**. All angles are in **radians**.



(a) Some trigonometry.

(i) Show that

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$\begin{aligned} (a+b)^5 &= \binom{5}{0} a^5 b^0 + \binom{5}{1} a^4 b^1 + \binom{5}{2} a^3 b^2 + \binom{5}{3} a^2 b^3 \\ &\quad + \binom{5}{4} a^1 b^4 + \binom{5}{0} a^0 b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \end{aligned}$$

Marks: 0, 4, 7, 10

(ii) Hence, using de Moivre's theorem show that

$$\cos(5x) = \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x$$

and using a well-known trigonometric formula, simplify the expression to

$$\cos(5x) = 16\cos^5 x - 20\cos^3 x + 5\cos x$$

$$\begin{aligned}
\cos(5x) + i\sin(5x) &= (\cos x + \sin x)^5 \\
&= \cos^5 5 + 5\cos^4 x(i\sin x) + 10\cos^3 x(i\sin x)^2 \\
&\quad +10\cos^2 x(i\sin x)^3 + 5\cos x(i\sin x)^4 + (i\sin x)^5 \\
&= \cos^5 5 + 5i\cos^4 x\sin x - 10\cos^3 x\sin^2 x \\
&\quad -10i\cos^2 x\sin^3 x + 5\cos x\sin^4 x + i\sin^5 x
\end{aligned}$$

$\cos(5x)$ corresponds to the real part so

$$\begin{aligned}
\cos(5x) &= \cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x sin^4 x \\
&= \cos^5 x - 10\cos^3 x\left(1-\cos^2 x\right) + 5\cos x\left(1-\cos^2 x\right)^2 \\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}$$

Marks: 0, 5, 7, 10, 15

(iii) Show that

$$16\cos^5\left(\frac{\pi}{10}\right) - 20\cos^3\left(\frac{\pi}{10}\right) + 5\cos\left(\frac{\pi}{10}\right) = 0$$

$$16\cos^5\left(\frac{\pi}{10}\right) - 20\cos^3\left(\frac{\pi}{10}\right) + 5\cos\left(\frac{\pi}{10}\right) = \cos\left(\frac{5\pi}{10}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

Marks: 0, 2, 3, 5

(b) Using the equation above, it is possible to prove that

$$\cos^2 \pi/10 = \frac{5+\sqrt{5}}{8}$$

(You do not have to prove this).

(i) Show that

$$\cos(2x) = 2\cos^2 x - 1$$

$$\cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x = \cos^2 x - \left(1-\cos^2 x\right) = 2\cos^2 x - 1$$

Marks: 0, 2, 3, 5

(ii) Hence, since

- $\pi/5 = 2 \times \pi/10$,

- $2\pi/5 = 2 \times \pi/5$,
- $4\pi/5 = 2 \times 2\pi/5$,

show that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4} \qquad \cos\left(\frac{4\pi}{5}\right) = \frac{-1-\sqrt{5}}{4}$$

(You must prove this and not only check the values with your calculator). Using the value and formula provided above

$$\begin{aligned}
\cos\left(\frac{\pi}{5}\right) &= \cos\left(2\frac{\pi}{10}\right) = 2\cos^2\left(\frac{\pi}{10}\right) - 1 = 2\left(\frac{5+\sqrt{5}}{8}\right) - 1 = \frac{1+\sqrt{5}}{4} \\
\cos\left(\frac{2\pi}{5}\right) &= \cos\left(2\frac{\pi}{5}\right) = 2\cos^2\left(\frac{\pi}{5}\right) - 1 = 2\left(\frac{1+\sqrt{5}}{4}\right)^2 - 1 = \frac{-1+\sqrt{5}}{4} \\
\cos\left(\frac{4\pi}{5}\right) &= \cos\left(2\frac{2\pi}{5}\right) = 2\cos^2\left(\frac{2\pi}{5}\right) - 1 = 2\left(\frac{-1+\sqrt{5}}{4}\right)^2 - 1 = \frac{-1-\sqrt{5}}{4}
\end{aligned}$$

Marks: 0, 5, 7, 10, 15

(c) Geometric approach

(i) The two points $A = (cos(2\pi/5), 0) = (\frac{-1+\sqrt{5}}{4}, 0)$ and $B = (cos(4\pi/5), 0) = (\frac{-1-\sqrt{5}}{4}, 0)$ form the diameter of a circle. Using these coordinates, determine the centre and radius of this circle. For the radius, use an expression of the form $(\frac{a+\sqrt{b}}{c})$ for your result, where a, b and c $\in \mathbb{Z}$.
Center (-1/4,0), Radius $\sqrt{5}/4$
Marks: 0, 2, 3, 5

(ii) Show that the radius of this circle is equal to the distance between the two points $C(-1/4, 0)$ and $D(0, 1/2)$

$$d = \sqrt{\left(-\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{4}$$

Marks: 0, 2, 3, 5

(iii) Aoife says that you can construct the star as follows.

- Draw a unit circle centred at the origin.
- Place the two points $C(-1/4, 0)$ and $D(0, 1/2)$.
- Draw the circle of radius CD centred at point C.
- The circle crosses the x axis at points A and B.
- Draw the lines perpendicular to the x axis passing through points A and B. They cross the unit circle at points J, K, L and M. Together with point $N(1, 0)$, they are the edges of the star.
- If the points are in the order J, K, L, M and N around the unit circle, join points J and L, K and M, and so on to form the star.

Explain why Aoife is correct and draw the star in the figure provided on the next page. You should detail every step of the construction.

The circle of radius CD cuts the x axis at the two points $((\sqrt{5}-1)/4, 0)$ and $((-\sqrt{5}-1)/4, 0)$ which correspond to $(\cos(2\pi/5), 0)$ and $(\cos(4\pi/5), 0)$. The two vertical lines will therefore cross the circle and the points corresponding to angles angles $2\pi/5$, $4\pi/5$, $6\pi/5$ and $8\pi/5$. Joining the points correctly leads to the star.

Marks: 0, 2, 3, 5

Figure

Marks: 0, 4, 7, 10