11 Induction and Binomial Theorem

11.1 Series Proofs

1. $1 + 2 + 3 + ... + n = \frac{n}{2}(n+1)$
2. $2 + 4 + 6 + ... + 2n = n(n+1)$
3. $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$
4. $1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{n}{3}(n+1)(n+2)$
5. $2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$
6. $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2}{4}(n+1)^2$
7. $1 + 2.2 + 3.2^2 + ...n2^{n-1} = (n-1)2^n + 1$
8. $\sum_{n=1}^{n} n(n+2) = \frac{n(n+1)(2n+7)}{6}$

11.2 Divisibility Proofs

1. Prove by induction that $11^n - 6$ is divisible by 5 for $n \in \mathbb{N}$
2. Prove by induction that $5^n - 1$ is divisible by 4 for $n \in \mathbb{N}$
3. Prove by induction that $3^{2n} - 1$ is divisible by 8 for $n \in \mathbb{N}$
4. Prove by induction that $8^n - 3^n$ is divisible by 5 for $n \in \mathbb{N}$
5. Prove by induction that $7^n + 4^n + 1$ is divisible by 6 for $n \in \mathbb{N}$
6. Prove by induction that $4^{n+1} + 5^{2n-1}$ is divisible by 21 for $n \in \mathbb{N}$
7. Prove by induction that $9^n - 1$ is divisible by 8 for $n \in \mathbb{N}$

11.3 Inequality Proofs

8. Prove by induction that $2^{n+1} > n^2$ for $n \in \mathbb{N}$
9. Prove by induction that $2^n > 2n$ for $n \geq 3, n \in \mathbb{N}$
10. Prove by induction that $n^2 > 2n + 3$ for $n \geq 4, n \in \mathbb{N}$
11. Prove by induction that $3^n > n^2$ for $n \geq 2, n \in \mathbb{N}$
12. Prove by induction that $n! > 2^n$ for $n \geq 4, n \in \mathbb{N}$
13. Prove by induction that $(n+1)! \geq 2^n$ for $n \in \mathbb{N}$

11.4 Binomial Theorem

1. Expand the following using the binomial theorem;
   (a) $(1+x)^4$
   (b) $(1+a)^5$

2. Write down the first three terms of the expansions of the following:
   (a) $(1+2x)^8$
   (b) $(a-3x)^5$
   (c) $(2a-b)^5$

3. What is the fifth term in the expansion of $(1-\frac{2}{x})^8$?

4. What is the middle term in the expansion of $(3-\frac{x}{3})^8$?

5. What is the coeffiecient of a^3 in the expansion of $(2+a)^5$?

6. What is the coeffiecient of x^4 in the expansion of $(2-\frac{x}{2})^5$?

7. Find the coeffiecient of x^5 in the expansion of $(2x^2+\frac{1}{x})^7$

11.5 Exam Questions

1. **2016 Paper 1**
   Prove by induction that $8^n - 1$ is divisible by 7 for all $n \in \mathbb{N}$

2. **2014 Paper 1**
   (a) Prove, by induction, that the sum of the first n natural numbers, $1+2+3+...+n$, is $\frac{n(n+1)}{2}$.
   (b) Hence, or otherwise, prove that the sum of the first n even natural numbers, $2+4+6+...+2n$, is n^2+n
   (c) Using the results from part form (a) and (b) above, find an expression for the sum of the first n odd natural numbers in its simplest form.

3. **2012 Paper 1**
   (a) Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:
   $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1-r^n)}{1-r}$
   (b) By writing the recurring parts as an infinite geometric series, express the following number as fraction of integers: $5.\dot{2}\dot{1} = 5.2121212121....$

4. **2014 Sample Paper 1**
   (a) i. Prove by induction that, for any n, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$
   ii. Find the sum of all the natural numbers from 51 to 100, inclusive.
   (b) Given that $p = log_c x$, express $log_c\sqrt{x} + log_c(cx)$ in terms of p.

11 Solutions

11.1 Proofs

1. NO SOLUTIONS AVAILABLE

11.4 Binomial

1. (a) $1 + 4x + 6x^2 + 4x^3 + x^4$
   (b) $1 + 5a + 10a^2 + 10a^3 + 5a^4 + a^5$
2. (a) $1 + 16x + 112x^2$
   (b) $a^5 - 15a^4x + 90a^3x^2$
   (c) $32a^5 - 80a^4b + 80a^3b^2$
3. $\frac{1120}{x^4}$
4. $70x^4$
5. 40
6. $\frac{5}{8}$
7. 560

11.5 Exam Questions

1. Prove
2. (a) Prove
   (b) Prove
   (c) n^2
3. (a) prove
   (b) $\frac{172}{33}$
4. (a) i. Prove
   ii. 3375
   (b) $\frac{3}{2}p + 1$