Leaving Certificate Examination, 2015

Sample paper prepared by Leamy Maths Community

Mathematics

Project Maths - Phase 3

Paper 2

Solutions

Higher Level

Sunday 26 April

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300 marks

Section A Concepts and Skills 150 Marks

Answer **all six** questions from this section.

Question 1 **(25 Marks)**

Line 1	$y = 2x + 1$
Line 2	$y = 3x - 2$
Line 3	$x + 3y = 4$
Line 4	$-2x + 4y = 4$
Line 5	$4x = 2y + 1$
Line 6	$x - y = 0$

(a) Which lines are parallel? Justify your answer.
Lines 1 and 5: they have the same slope s=2

(b) Which two lines are perpendicular? Justify your answer.
Lines 2 and 3 the product of their slopes is -1.
5 marks

(c) Calculate the equation of the lines which make a 45° angle with Line (1) and has the same y intercept as Line (4).
The slopes of the lines verify

$$\tan 45 = \pm\frac{m-2}{1+2m} \iff 1 + 2m = m - 2 \qquad d - 1 - 2m = m = 2$$
$$\iff m = \frac{1}{3} \qquad m = -3$$

The y-intercept is 1 so the two equations are

$$y = \frac{1}{3}x + 1 \qquad y = -3x + 1$$

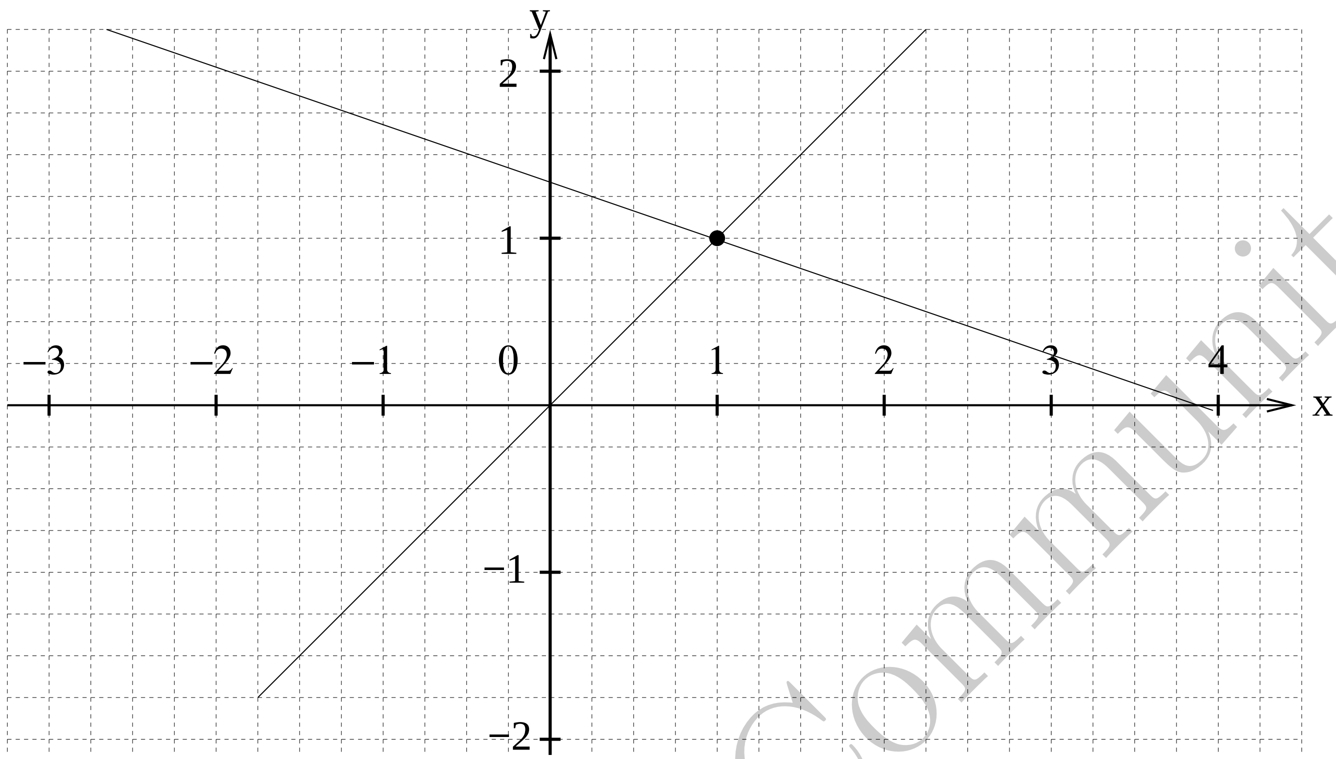
10 marks

(d) Plot Lines (3) and (6) on the diagram below and read on the graph where they cross. The two lines cross at (1,1)
5 marks

(e) Calculate the intersection point between Line (3) and Line (6) analytically.

$$x + 3y = 4 \qquad -x + y = 0 \Longrightarrow x = 1 \qquad y = 1$$

The two lines cross at (1,1)
5 marks



Question 2 **(25 Marks)**

(a) Calculate the equation of the circle $(\mathcal{C}_1)$ of radius $R = 5$ and centre $(-3, -2)$.

$$(x+3)^2 + (y+2)^2 = 5^2 = 25 \Longrightarrow x^2 + 6x + y^2 + 4y - 12 = 0$$

5 marks

(b) Determine the centre and radius of the circle $(\mathcal{C}_2)$ described by the following equation:

$$x^2 + y^2 - 18x - 14y + 30 = 0$$

$$x^2 + y^2 - 18x - 14y + 30 = 0 \Longrightarrow (x-9)^2 + (y-7)^2 = 100$$

Centre(9,7), Radius=10
5 marks

(c) Show that the two circles $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ are externally touching.
Distance between the two centres is $d = \sqrt{(9+3)^2 + (7+2)^2} = 15 = 10 + 5$. The distance centres is the sum of the radius: the circles are touching externally.
7 marks

(d) Calculate the coordinates of the point common to both circles $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$
The line joining the 2 centres is $4y - 3x = 1$. Subtracting the equations of the 2 circles leads to $24x + 18y = 42$. Solving the system leads to $x = 1$ and $y = 1$. The common point is (1,1).
8 marks

Question 3 **(25 Marks)**

In a school, a sample of 50 randomly selected students found that on average, they live 24 minutes drive away from the school with a 4 minute standard deviation.

(a) Construct a 95% confidence interval for the driving time from home to school the students of this school.

$$CI = 24 \pm 1.96\frac{4}{\sqrt{50}} = [22.9;\ 25.1]$$

6 marks

(b) The school principal believes that on average, students live 20 minute drive away from the school. Test this hypothesis using a 5% level of significance. Detail all steps of the method
$H_0:\ \mu = 20$ and $H_1:\ \mu \neq 20$.
The mean is outside the confidence interval, H_0 should be rejected. You are 95% confident that the mean is not 20, the students do not leave on average 20 minutes away from the school
6 marks

(c) 50 sixth year students were asked if they had found a summer job. 20 students answered yes. Construct a 95% confidence interval for the proportion of students who will have a summer job.

$$p_0 = \frac{20}{50} = 0.4$$
$$CI = 0.4 \pm 1.96\sqrt{\frac{(0.4)(1-0.4)}{50}} = [0.264; 0.536]$$

7 marks

(d) The school principal believes that on average, half the sixth year students will be working during the summer. Test this hypothesis using a 5% level of significance. Detail all steps of the method
$H_0:\ p = 0.5$ and $H_1:\ p \neq 0.5$.
The mean is inside the confidence interval so you accept H_0: you are 95% confident that the half the sixth year students will be working during the summer.
6 marks

Question 4 **(25 Marks)**

A screw manufacturer sells its products plastic bags. The number of screws per bag follows a normal distribution. On average, there are 150 screws per bag with a standard deviation of 9.

(a) A bag is chosen randomly

(i) Calculate the probability that the bag contains less than 155 screws

$$P(x \leq 155) = P\left(Z \leq \frac{155-150}{9}\right) = P(Z \leq 0.56) = 0.7123$$

The probability is 0.7123
3 marks

(ii) What is the probability to find between 140 and 160 screws

$$\begin{aligned} P(140 \leq X \leq 160) &= P\left(\frac{140-150}{9} \leq Z \leq \frac{160-150}{9}\right) \\ &= P(-1.11 \leq Z \leq 1.11) \\ &= P(Z \leq 1.11) - P(Z \leq -1.11) \\ &= 2P(Z, 1.11) - 1 = 0.733 \end{aligned}$$

Probability is 0.733
5 marks

(iii) Can you find more than 180 screws in a bag? Justify your answer.

$$P(X \geq 180) = P\left(Z \geq \frac{180-150}{9}\right) = P(Z \geq 3.33) = 1 - P(Z \leq 3.33) \approx 0$$

This is nearly impossible
3 marks

(b) 2% of the screws are defective. 10 screws are chosen randomly

(i) Calculate the probability that a single screw is defective

$$P(X = 1) = \mathcal{C}_1^{10} 0.02^1 0.98^9 = 0.1667$$

There is 16.7% chance that one screw is defective
5 marks

(ii) What is the probability that 2 screws or more are defective?

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \mathcal{C}_0^{10} 0.02^0 0.98^1 0 - \mathcal{C}_1^{10} 0.02^1 0.98^9 \\ &= 0.0162 \end{aligned}$$

There is 1.62% chance that at least two screws are defective
9 marks

Question 5 **(25 Marks)**

(a) Calculate the solutions of the following equations

(i) $x \in [0,\ 2\pi]$

$$cosx = -\frac{1}{2}$$

$$x = \frac{2\pi}{3} + 2\pi n \qquad x = -\frac{2\pi}{3} + 2\pi n \qquad n \in \mathbb{Z}$$

The two possible solutions in $[0,\ 2\pi]$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$
3 marks

(ii) $x \in [-2\pi,\ 2\pi]$

$$\sin(3x) = \frac{\sqrt{3}}{2}$$

$$3x = \frac{\pi}{3} + 2\pi n \qquad 3x = \frac{2\pi}{3} + 2\pi n \qquad n \in \mathbb{Z}$$
$$x = \frac{\pi}{9} + \frac{2\pi n}{3} \qquad x = \frac{2\pi}{9} + \frac{2\pi n}{3} \qquad n \in \mathbb{Z}$$

The solutions are $-\frac{17\pi}{9}$, $-\frac{16\pi}{9}$, $-\frac{11\pi}{9}$, $-\frac{10\pi}{9}$, $-\frac{5\pi}{9}$, $-\frac{4\pi}{9}$, $\frac{\pi}{9}$, $\frac{2\pi}{9}$, $\frac{7\pi}{9}$, $\frac{8\pi}{9}$, $\frac{13\pi}{9}$ and $\frac{14\pi}{9}$

7 marks

(b) Trigonometry equation

(i) Show that

$$2\sin(2x)\sin x - \cos x = \cos x\left(3 - 4\cos^2 x\right)$$

$$\begin{aligned}
2\sin(2x)\sin x - \cos x &= 4(\cos x \sin x)\sin x - \cos x \\
&= 4\cos x \sin^2 x - cosx \\
&= 4\cos x\left(1 - \cos^2 x\right) - cosx \\
&= \cos x\left[3 - 4\cos^2 x\right]
\end{aligned}$$

5 marks

(ii) Solve the equation

$$2\sin(2x)\sin x = \cos x$$

$$\begin{aligned}
2\sin(2x)\sin x = \cos x &\iff \cos x\left(3 - 4\cos^2 x\right) \\
&\iff \cos x\left(\sqrt{3} - 2\cos x\right)\left(\sqrt{3} + 2\cos x\right) = 0 \\
&\iff \cos x = 0 \quad \cos x = \frac{\sqrt{3}}{2} \quad \cos x = -\frac{\sqrt{3}}{2} \\
&\iff x = \pm\frac{\pi}{2} + 2k\pi \quad x = \pm\frac{\pi}{6} + 2k\pi \quad x = \pm\frac{5\pi}{6} + 2k\pi \quad k \in \mathbb{Z}
\end{aligned}$$

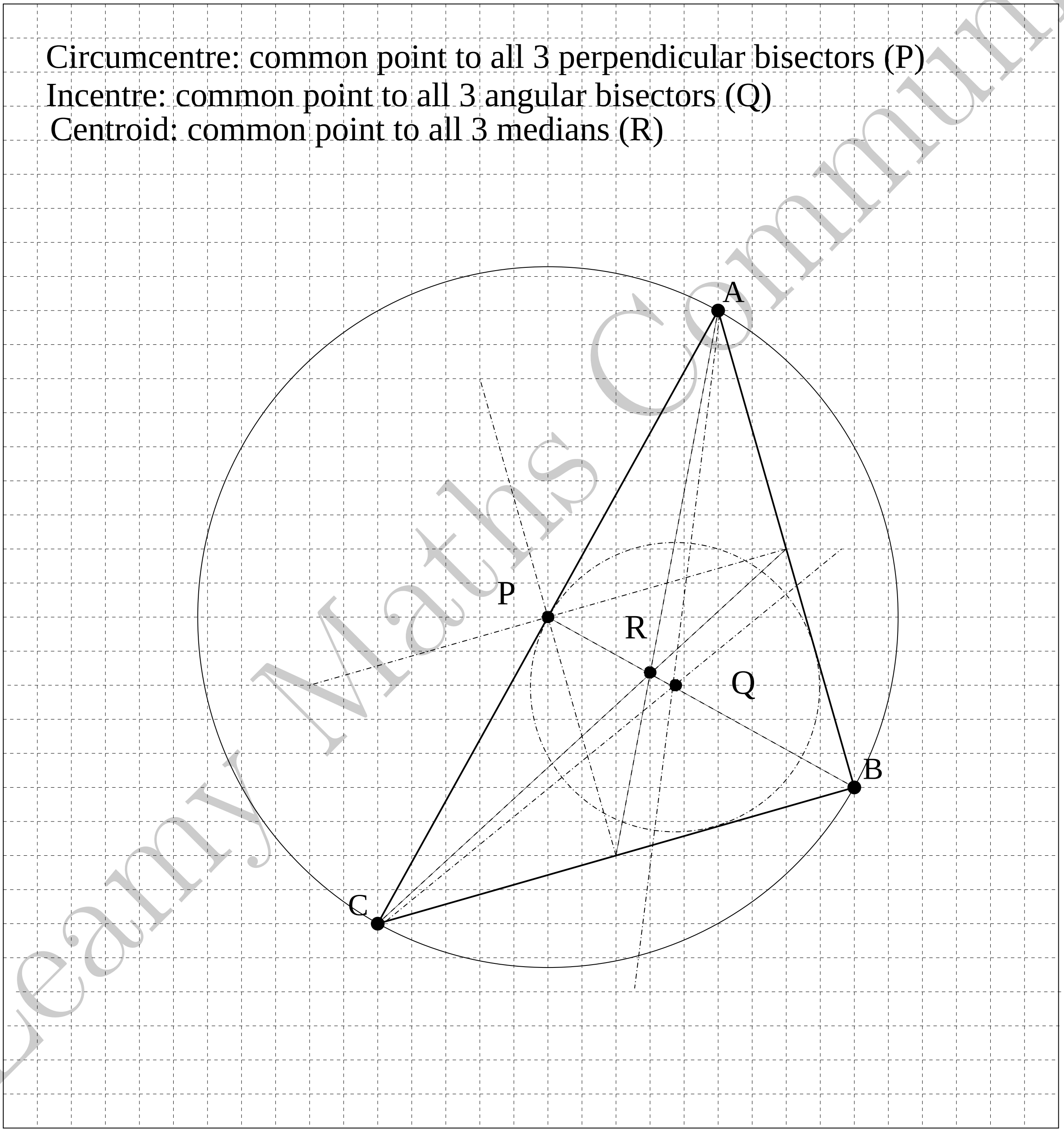
10 marks

Question 6 **(25 Marks)**

Answer **either** 6A **or** 6B.

Question 6A

Define the circumcentre, the incentre and the centroid of a triangle. Construct these 3 points for the triangle below without using a graduate ruler or a set square.



Question 6B

Points A, B, C, and D are on a circle, see figure below, and $< ADB = 45°$. The lines (AD) and (BC) cross at point E. Show that if (AD) and (BC) are perpendicular and $|AB| = |AC|$, E is the centre of the circle. (Hint: compare the angles $< ACB$ and $< ADB$ and calculate other angles.)

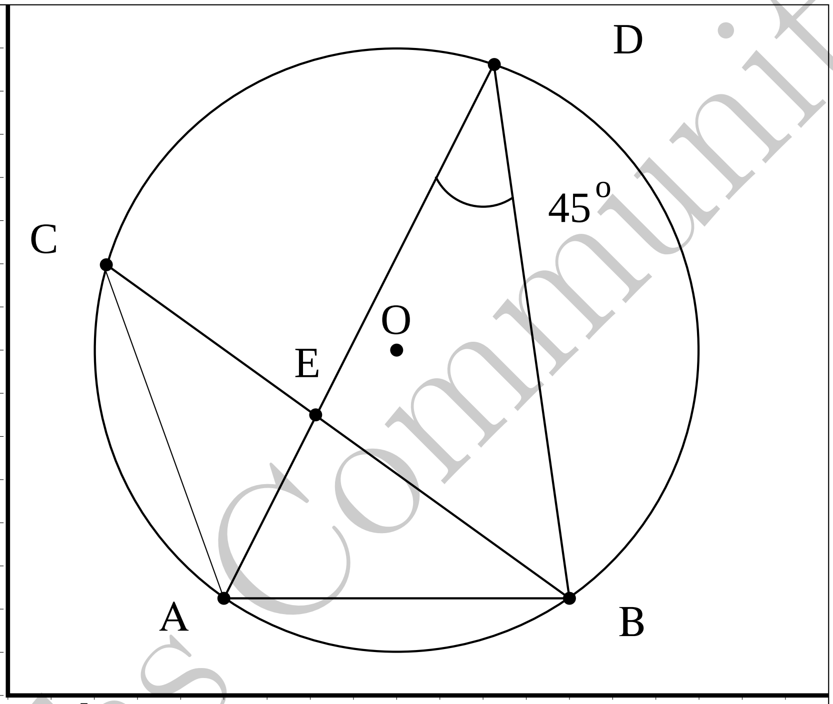
<ABD=<ACB=45
AB=AC so <ABC=45

<BAC=180−<ACB−<BAC
=90

This means [BC] is a diameter of the circle

<EBD=180−<BED−<BDE
=45

<ABD=<ABE+<EBD=90



so [AD] is a diameter of the circle

E is the intersection of two diameters of the circle so this is the centre of the circle

Section B Contexts and Applications 150 Marks

Answer Question 7 **and** Question 8.

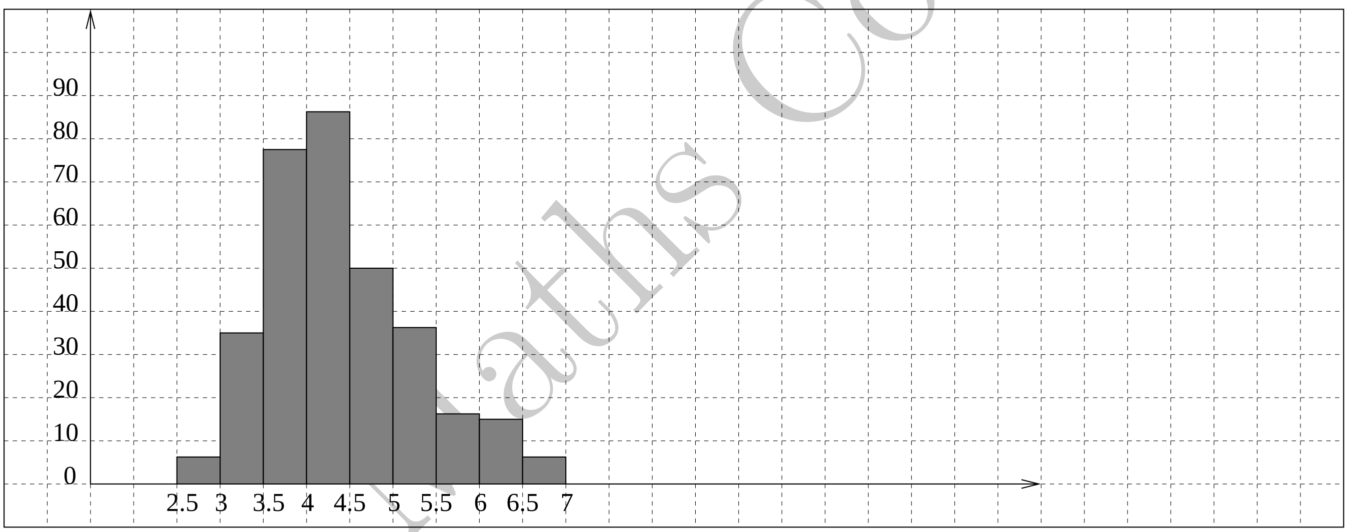
Question 7 **(75 Marks)**

The table below shows running times at the 2014 Dingle marathon.

Running time (hours)	2.5 – 3	3 – 3.5	3.5 – 4	4 – 4.5	4.5 – 5	5 – 5.5	5.5 – 6	6 – 6.5	6.5 – 7
Number of runners	6	35	78	87	50	36	17	15	6

(a) Characteristics of the distribution

(i) Create a suitable representation of the running time for the 2014 marathon.



5 marks

(ii) Describe the distribution. This should include information such as shape, range, estimation for the median value. What is the position of the mean relative to the median? Can you define a mode?

Distribution is right skewed (positively skewed). Distribution quickly rises from 0 below 2.5 to reach its peak for t between 4 and 4.5. Then it decreases progressively to reach 0 after t=7. The median is smaller than the mean. Here the mode can not be defined as the data is more or less continuous. The median is between 4 and 4.5.

5 marks

(iii) Is it possible to use a standard normal distribution to represent the data? Justify your answer

The curve is not symmetrical and has a long right tail. a standard distribution would not suit the data.

5 marks

(iv) Using mid interval values, estimate the average running time.
E=4.4≈ 4 hours and 24 minutes
5 marks

(b) The organisers of the race choose 10 random runners to quickly evaluate the average running time of participants. They find an average value of 4.75 hours and a standard deviation of 0.7 hours.

(i) Construct a 95% confidence interval for the average running time of this race according to this sample.

$$CI = 4.75 \pm 1.96\frac{0.7}{\sqrt{10}} = [4.32; 5.18]$$

10 marks

(ii) The actual average running time is actually 4.4 hours. The organisers say that their sample was representative of all runners in the race as far as the running time is concerned. Test this hypothesis using a 5% level of significance. Detail all steps of the method.
$H_0: \ \mu = 4.4$ $H_1: \ \mu \neq 4.4$
4.4 is in the confidence interval so you accept H_0: you are 95% confident that the average running time is 4.4 hours.
10 marks

(c) The organisers want 10 values to estimate the race running time. 5% of runners abandon the race and do not complete the marathon. The organisers therefore want to choose more than ten people to make sure they have a sufficient number of data to estimate the average running time. They decide to take a sample of 12 people.

(i) Can whether or not a person finishes the race be considered a Bernoulli trial? You should specify the conditions for Bernoulli trials (also called binomial trials) and then conclude if it is possible to use this method here.

- Success/Failure
- Finite number of trials
- Independent trials
- Constant probabilities

All 4 are verified here.
5 marks

(ii) What is the probability that exactly 10 people finish the race?

$$P(X = 10) = \mathcal{C}_{10}^{12} 0.95^{10} 0.05^2 = 0.099$$

Probability is 9.9%
5 marks

(iii) What is the probability that 10 or more people finish the race?

$$\begin{aligned} P(X \geq 10) &= P(X = 10) + P(X = 11) + P(X = 12) \\ &= \mathcal{C}_{10}^{12} 0.95^{10} 0.05^2 + \mathcal{C}_{11}^{12} 0.95^{11} 0.05^1 + \mathcal{C}_{12}^{12} 0.95^{12} 0.05^0 \\ &= 0.099 + 0.341 + 0.540 = 0.98 \end{aligned}$$

The probability is 98%
7 marks

(d) The organisers want to study the average running time in the sample. The following probabilities should be calculated using the table provided (these runners have completed the race).

(i) What is the probability that a randomly selected runner finishes the race between 3 and 4 hours?

$$p = \frac{35 + 78}{330} = 0.342$$

The probability is 34.2%
5 marks

(ii) What is the probability that 3 randomly selected runners all finish the race between 3 and 4 hours?

$$p = \frac{\mathcal{C}_3^{113}}{\mathcal{C}_3^{330}} = 0.0394$$

The probability is 3.94%
5 marks

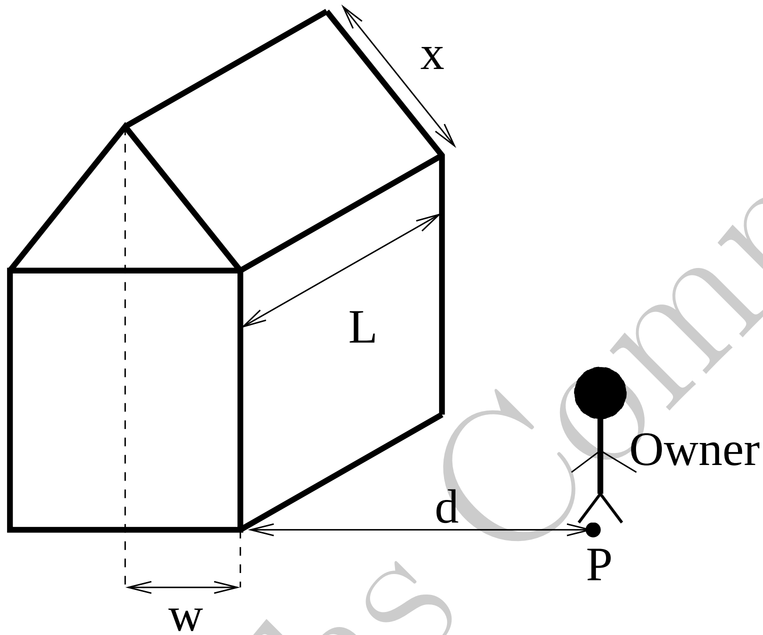
(iii) If two runners are randomly chosen, what is the probability that 1 runner finishes between 3 and 4 hours and 1 runner finishes between 4 and 5 hours?

$$p = \frac{2 \times 113 \times 137}{330 \times 329} = 0.285$$

The probability is 28.5%
8 marks

Question 8 **(75 Marks)**

(a) A house is to become eco-friendly. The roof is going to be covered with solar panels and rain collectors will be installed. The owner wants to estimate the area of the roof without climbing on top of the building but is happy to do all the necessary measurements on the ground. He uses a clinometer to calculate the angles of elevation from point P on the ground to the bottom and top of the roof and finds $45°$ to the bottom of the roof and $52.12°$ to the top of the roof.



The owner measured the following: $w = 4$m, $L = 10$m and $d = 10$m. The dimension x should be calculated correct to one decimal place.

(i) Explain how to use a clinometer.

- Measure the distance between the building and the observer.
- Look at the top of the building
- Read the value of the angle on the scale of the clinometer.
- Use trigonometry to calculate the length of interest.

5 marks

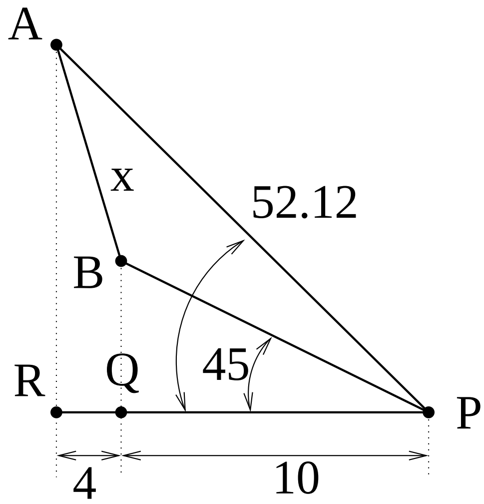
(ii) Calculate the area of the roof correct to one decimal place.

Using the measured angles, one gets

$$\cos 45 = \frac{PQ}{BP} \implies BP = \frac{10}{\cos 45} = 14.1\text{m}$$
$$\cos 52.12 = \frac{PR}{AP} \implies AP = \frac{14}{\cos 52.12} = 22.8\text{m}$$

$< APB = 52.12 - 45 = 7.12°$. Using the cosine rule

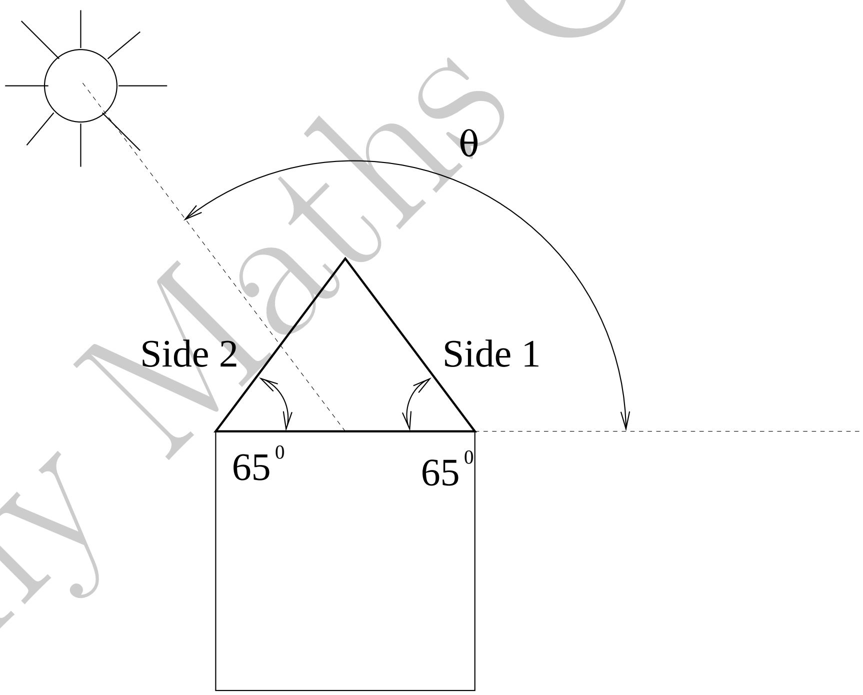
$$x = \sqrt{14.1^2 + 22.8^2 - 2 \times 14.1 \times 22.8 \cos 7.12} \approx 9\text{m}$$



The area of the roof is therefore

$$Area = 2 \times x \times L = 180\text{m}^2$$

10 marks



(b) The left side of the house is facing West, the right side is facing East. It is assumed that there are 12 hours of light every day from 6 in the morning till 6 in the evening. At sun rise, $\theta = 0$, at sun set, $\theta = 180$.

(i) For what values of θ will side 1 face the sun ?

$$0 \leq \theta \leq 115$$

(ii) For what values of θ will side 2 face the sun?

$$65 \leq \theta \leq 180$$

5 marks

(iii) Assuming the sun moves 15° per hour, estimate from what time to what time each side of the roof will be lit.
Side 1: 6.00 to 13.40
Side 2: 10.20 to 18.00
5 marks

(iv) The efficiency of the solar panels is maximum when the sun shines perpendicularly to the roof. Identify which expression represent the efficiency on side 1 and side 2 and for both case, specify for which values of θ it may be used (the expressions are in degrees).

$$A: \quad E = \cos(\theta - 25) \qquad\qquad B: \quad E = \cos(\theta - 155)$$

Side 1 is A, Side 2 is B
5 marks

(v) Calculate for which values of the angle θ the efficiency is equal to 50% (E=0.5). Perform the calculation for both sides.
Side 1: $\cos(\theta - 25) = 0.5$ leads to $\theta - 25 = \pm 60$ so $\theta = -35$ (impossible) or $\theta = 85$
Side 2: $\cos(\theta - 155) = 0.5$ leads to $\theta - 155 = \pm 60$ so $\theta = 215$ (impossible) or $\theta = 95$
8 marks

(vi) Using the results of the previous question, determine for what time of the day the efficiency of the solar panel will exceed 50% for side 1 and for side 2.
Side 1: 6.00 to 11.40
Side 2: 12.20 to 18.00
7 marks

(c) Rain is falling vertically at a rate of 5mm per hour per square metre. A rain collector gathers of all the water falling on the roof. You want to calculate the volume of water falling on the roof. You therefore need to use the catchment area (area in the direction perpendicular to the rain direction).

(i) Explain why the catchment area of the house must be 80m^2 and not the answer you found in part a (ii).
Looking from the sky, the house looks like a rectangle 10m by 8 m
5 marks

(ii) It rains for 2 hours. What volume of water fell on the roof during this time (in m^3)?

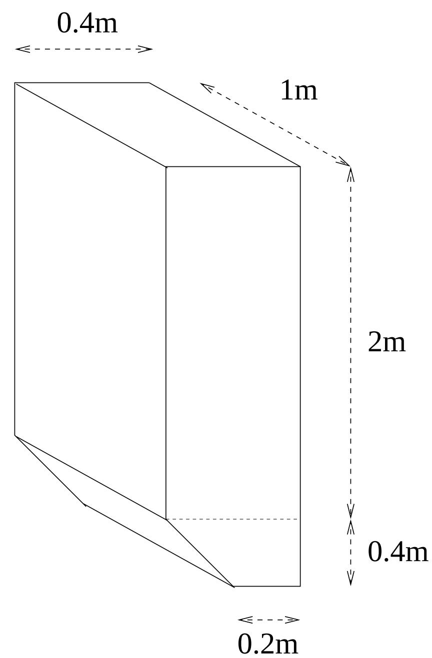
$$= V = 80 \times 0.005 \times 2 = 0.8m^3$$

5 marks

(iii) The rain collector has the following characteristics
What is the maximum volume of water contained in this rain collector?

$$V = \frac{0.4 + 0.2}{2} \times 0.4 \times 1 + 2 \times 1 \times 0.4 = 0.92m^3$$

5 marks



(d) What is the level of water in the tank if 0.8m^3 of rain fell on the roof?
Volume in the bottom part: $0.5 \times (0.2 + 0.4) \times 0.4 \times 1 = 0.12m^3$
Volume in the rectangualr part: $0.8 - 0.12 = h \times 0.4 \times 1 \Longrightarrow h = 0.64/0.4 = 1.7m$
Height is 0.4+1.7=2.1m
5 marks

(e) How much longer should it rain at this rate to fill the tank?
Volume left is 0.92-0.8=0.12m^3. If it rains for t hours

$$t \times 0.005 \times 80 = 0.12 \Longrightarrow t = 0.3h$$

It should rain for 18 more minutes (=0.3 hours)
5 marks