Leaving Certificate Examination, 2019

Sample paper prepared by Leamy Maths Community

Mathematics

Paper 2

Higher Level

28 April 2019

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LEAMY MATHS COMMUNITY

Solutions

300 marks

Sample Instructions

There are two sections in this examination paper:

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer questions as follows:

In Section A, answer all six questions.
In Section B, answer all three questions.

Write your answers in the spaces provided in this booklet. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the booklet of *Formulae and Tables*. You must return it at the end of the examination.

You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	150 Marks

Answer **all six** questions from this section.

Question 1 (25 Marks)

Consider the line $\mathcal{L}_1 : y = \frac{4x-1}{3}$

(a) Check that the line $\mathcal{L}_1$ contains the two points $A(1,1)$ and $B(4,5)$.

$$\frac{4(1)-1}{3} = \frac{3}{3} = 1$$
$$\frac{4(4)-1}{3} = \frac{15}{3} = 5$$

so the two points $A(1,1)$ and $B(4,5)$ are on the line.
Marks: 0, 3, 5

(b) Calculate the distance between line $\mathcal{L}_1$ and point $C(-6,0)$.

$$d = \frac{|4(-6)-3(0)-1|}{\sqrt{3^2+4^2}} = \frac{25}{5} = 5$$

Marks: 0, 3, 5

(c) Calculate the equation of line $\mathcal{L}_2$, the line parallel to line $\mathcal{L}_1$ which contains point C. Give the equation in the form $ax + by + c = 0$ where a, b, c in $\mathbb{Z}$.

$$4x - 3y + 24 = 0$$

Marks: 0, 3, 5

(d) Calculate the equation of line $\mathcal{L}_3$, the line perpendicular to line $\mathcal{L}_1$ which contains point C. Give the equation in the form $ax + by + c = 0$ where a, b, c in $\mathbb{Z}$

$$3x + 4y + 18 = 0$$

Marks: 0, 3, 5

(e) Calculate the coordinates of point D the intersection between lines $\mathcal{L}_1$ and $\mathcal{L}_3$ and the distance $|CD|$
The two lines cross at $D(-2,-3)$. $|CD| = 5$
Marks: 0, 3, 5

Question 2 **(25 Marks)**

(a) Find the radius and centre of the two circles

$$\mathcal{C}_1 \qquad x^2 - 8x + y^2 + 12y - 48 = 0$$
$$\mathcal{C}_2 \qquad x^2 - 14x + y^2 + 20y + 124 = 0$$

$C_1 = (4, -6),\ R_1 = \sqrt{48 + 16 + 36} = 10$
$C_2 = (7, -10),\ R_2 = \sqrt{-124 + 49 + 100} = 5$
Marks: 0, 3, 5

(b) Show that the two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are internally touching.
$|C_1C_2| = \sqrt{3^2 + 4^2} = 5 = R_2 - R_1$ so the two circles are touching internally.
Marks: 0, 3, 5

(c) Calculate the equation of the tangent common to both circles $\mathcal{C}_1$ and $\mathcal{C}_2$. Give the equation in the form $ax + by + c = 0$ where a, b, c in $\mathbb{Z}$.

$$6x - 8y - 172 = 0 \Longrightarrow 3x - 4y - 86 = 0$$

Marks: 0, 3, 5

(d) Find the coordinates of the common point to circles $\mathcal{C}_1$ and $\mathcal{C}_2$.
result from the previous question and line joining the two centres

$$slope = \frac{-6+10}{4-7} = -\frac{4}{3} \Longrightarrow y = -\frac{4}{3}(x-4) - 6 \Longrightarrow 4x + 3y = -2$$

Common point is (10, -14)

Marks: 0, 3, 5, 8, 10

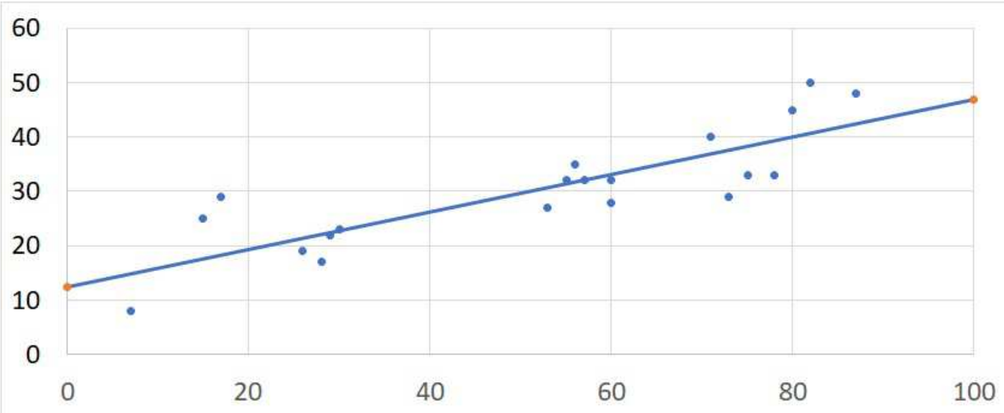
Question 3 **(25 Marks)**

The price of an item of clothing is compared to the time it takes to produce it.

Item	1	2	3	4	5	6	7	8	9	10
Time (min)	53	73	60	60	17	78	30	29	87	71
Price (€)	27	29	32	28	29	33	23	22	48	40

Item	11	12	13	14	15	16	17	18	19	20
Time (min)	55	80	15	26	75	7	82	57	56	28
Price (€)	32	45	25	19	33	8	50	32	35	17

(a) Draw a scatter diagram of the data



Marks: 0, 3, 5

(b) Draw the line of best fit on the graph above and calculate its equation.

$$y = 12.373 + 0.346x$$

Marks: 0, 3, 5, 8

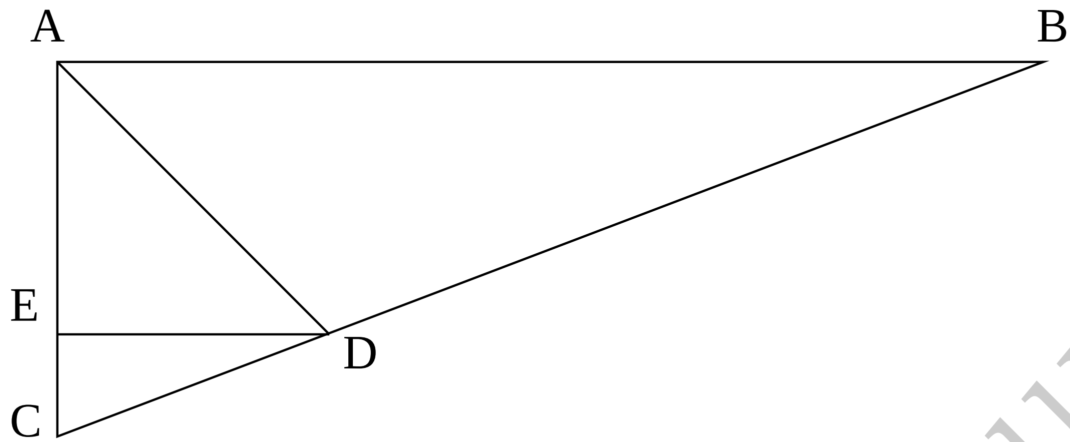
(c) Calculate the correlation coefficient. Interpret the value and explain how this can be related to the scatter diagram.
$\rho = 0.84$. This is quite high. The points are close to the line of best fit.
Marks: 0, 3, 5

(d) Calculate the price of items requiring production times of 50 minutes and 240 minutes. Which price is the most reliable. Justify your answer.
50 minutes leads to €29.67, this is in range so is quite reliable
240 minutes leads to €95.41, this is not reliable as the number of minutes is completely out of range.
Marks: 0, 3, 5, 8

Question 4 (25 Marks)

The figure above has the following characteristics:

- $< EAD = 45°$, $< ACD = 76°$, $< ABC = 14°$, $< AED = 90°$
- $|AD| = 4\sqrt{2}$, $|CE| = 1$



(a) Show that the two triangles ABC and CDE are similar.

$$< CED = 90^\circ \quad < CAB = 180 - 76 - 14 = 90^\circ$$
$$< ACB =< ECD$$

You must prove that $< CAB = 90^\circ 4$ as this is not given.

Since the sum of the angles in a triangle is 180°, $< ABC =< EDC$. The three sets of angles are equal so the triangles are similar.
Marks: 0, 3, 5

(b) Calculate the two distances $|AB|$ and $|BD|$. Give your results correct to two decimal places.

$$|AE| = 4 \quad |DE| = 4 \quad |AC| = 5$$
$$\frac{|CE|}{|AC|} = \frac{|DE|}{|AB|} \Longrightarrow \frac{1}{5} = \frac{4}{|AB|} \Longrightarrow |AB| = 20$$
$$\frac{|AD|}{\sin < ABD} = \frac{|BD|}{\sin < DAB} \Longrightarrow \frac{4\sqrt{2}}{\sin 14} = \frac{BD}{\sin 45} \Longrightarrow |BD| = 16.53$$

Marks: 0, 3, 5, 8, 10, 12, 15

(c) Calculate the area of the triangle ABD. Give your results correct to two decimal places.

$$Area = \frac{1}{2}|AB|h = \frac{1}{2}|AB| \times |AE| = \frac{(20)(4)}{2} = 40$$

Marks: 0, 3, 5

Question 5 (25 Marks)

(a) Show that $\cos(2x) = 2\cos^2 x - 1$. Hence show that

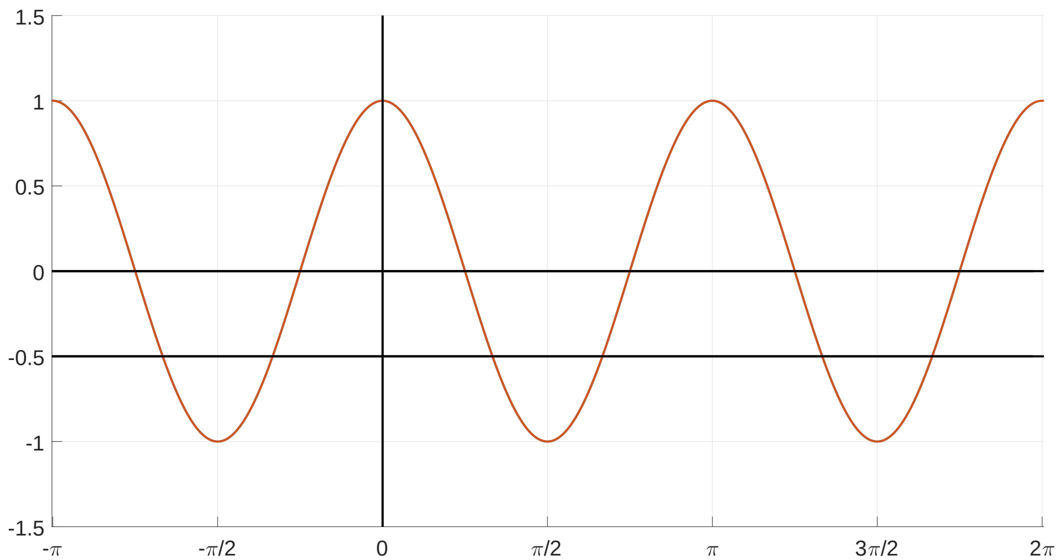
$$\cos(4x) = 8\cos^4 x - 8\cos^2 x + 1$$

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - \left(1 - \cos^2 x\right) = 2\cos^2 x - 1$$
$$\cos(4x) = 2\cos^2(2x) - 1 = 2((2\cos^2 x) - 1)^2 - 1 = 8\cos^4 x - 8\cos^2 x + 1$$

Marks: 0, 3, 5, 8, 10

(b) Plot the line $y = -\frac{1}{2}$ and the function $f(x) = \cos 2x$ on the graph below using the table below and the properties of the cos function.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$
f(x)	1	$\frac{\sqrt{2}}{2}$	0	-$\frac{\sqrt{2}}{2}$	-1	-$\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$



Marks: 0, 3, 5

(c) Solve the equation

$$\cos 2x = -\frac{1}{2} \qquad x \in [0; 2\pi]$$

Explain how the solutions can be read from the graph above.

$$2x = \frac{2\pi}{3} \implies x = \frac{\pi}{3} \quad x = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$
$$2x = \frac{4\pi}{3} \implies x = \frac{2\pi}{3} \quad x = \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$$

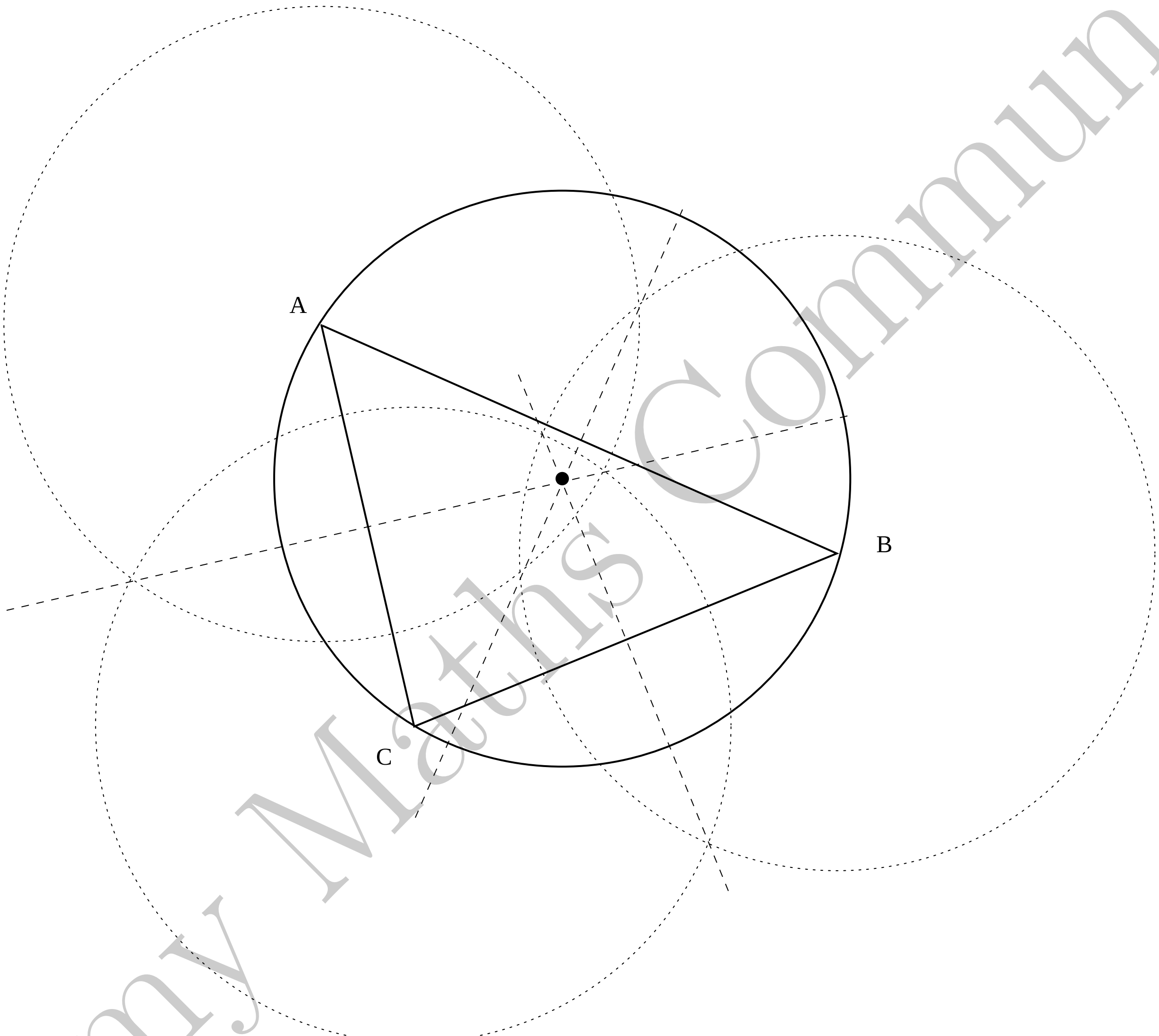
This is where the horizontal line $y = -\frac{1}{2}$ crosses the curve.

Marks: 0, 3, 5, 8, 10

Question 6 (25 Marks)

(a) Construct the circumcentre and draw the circumcircle for triangle ABC using your ruler and compass.

Marks: 0, 3, 5, 8, 10



(b) You have a pyramid with a square base, a cone and a cylinder.

(i) If the three have the same base area and the same volume, order their heights in ascending order.

$$\begin{array}{ll} \textit{Pyramid} & V = \frac{1}{3}hS \Longrightarrow h_{Py} = 3\frac{V}{S} \\ \textit{Cylinder} & V = hS \Longrightarrow h_{Cy} = \frac{V}{S} \\ \textit{Cone} & V = \frac{1}{3}hS \Longrightarrow h_{Co} = 3\frac{V}{S} \end{array}$$

Order: Cylinder-Cone-Pyramid or Cylinder-Pyramid-Cone
Marks: 0, 3, 5

(ii) If the three have the same base perimeter and the same volume, order their heights in ascending order.

$$\begin{array}{ll} \textit{Pyramid} & p = 4a \Longrightarrow S = a^2 = \frac{p^2}{16} \Longrightarrow V = \frac{1}{3}hS = \frac{hp^2}{48} \Longrightarrow h_{Py} = 48\frac{V}{p^2} \\ \textit{Cylinder} & p = 2\pi r \Longrightarrow r = \frac{p}{2\pi} \Longrightarrow A = \pi r^2 = \frac{p^2}{4\pi} \Longrightarrow V = h\frac{p^2}{4\pi} \Longrightarrow h_{Cy} = 4\pi\frac{V}{p^2} \\ \textit{Cone} & p = 2\pi r \Longrightarrow r = \frac{p}{2\pi} \Longrightarrow A = \pi r^2 = \frac{p^2}{4\pi} \Longrightarrow V = \frac{h}{3}\frac{p^2}{4\pi} \Longrightarrow h_{Co} = 12\pi\frac{V}{S} \end{array}$$

Order: Cylinder-Cone-Pyramid
Marks: 0, 3, 5, 8, 10

Section B Contexts and Applications 150 Marks

Answer **all three** questions from this section.

Question 7 (50 Marks)

When you play the Euromillion, you select 5 main numbers between 1 and 50 and two lucky stars between 1 and 12. You become a winner when at least two of the main numbers you selected are in the draw. The more numbers and the more lucky stars you have in common with the draw, the larger your prize. You win the jackpot if you choose the correct 5 main numbers and the two lucky stars.

(a) Probability of winning the jackpot

(i) Calculate the number of ways of selecting 5 numbers among 50.

$$C_5^{50} = 2118760$$

(ii) Calculate the number of ways of selecting 2 numbers among 12.

$$C_2^{12} = 66$$

Marks: 0, 3, 5

(iii) Hence calculate the probability of winning the jackpot. Give your result in the form $a \times 10^{-9}$ where $1 \leq a < 10$.

$$p = \frac{1}{66 \times 2118760} = 7.15 \times 10^{-9}$$

Marks: 0, 3, 5

(b) Playing one line in the Euromillion costs €2.5

(i) How much much will you have to pay if you want to be sure to win the jackpot (i.e. if you play every single combination)?

$$Pay = 2.5 \times 66 \times 2118760 = 349595400$$

(ii) The maximum possible prize at the Euromillion is €190 million: can you interpret this value in terms of your answer to **b(i)**?
The value is significantly lower than the value above: players can not beat the lottery.
Marks: 0, 3, 5

The last time there was an Irish winner, the money prizes were as follows

Match	Probability	Prize (€)
5 + 2 Stars	7.15×10^{-9}	175,475,380
5 + 1 Stars	1.43×10^{-7}	317,493
Match 5	3.22×10^{-7}	41,082
4 + 2 Stars	1.61×10^{-6}	2,349
4 + 1 Stars	3.22×10^{-5}	145
3 + 2 Stars	7.08×10^{-5}	84
Match 4	7.24×10^{-5}	54
2 + 2 Stars	0.001015	15
3 + 1 Stars	0.001416	12
Match 3	0.003186	11
1 + 2 Stars	0.00533	8
2 + 1 Stars	0.02029	7
Match 2	0.04566	4

(iii) Calculate the expected pay-out for an average player. Give your results correct to 3 decimal places.

$$E = 1.767$$

The average player wins €1.767
Marks: 0, 3, 5

(iv) 40 million grids were played on that day, each for a cost of €2.5. Based on the expected payout calculated in the previous section, how much money is there left for the organisers?

$$E = 40000000(2.5 - 1.767) = 29322611$$

On average, the lottery has €29,322,611 (Note that this is an average if there were many draws with this payout profile. In this specific case the top payout is more than the amount played so the lottery would loose money on this draw but it would have a lot of money left from previous draws as this is a rollover)
Marks: 0, 3, 5

(c) Barry's favourite number for the lucky stars is number 5. Two lucky star numbers are drawn between 1 and 12. Show that the probability that 5 is one of the two lucky stars is p=1/6.

$$p = \frac{C_{11}^1 C_1^1}{C_{12}^2} = \frac{1}{6}$$

Marks: 0, 3, 5

(d) Bary carries out a probability study on his lucky star number. He wants to know how many times his lucky star number is likely to be drawn in the next ten Euromillion draws.

(i) State the 4 conditions to be verified for using the the binomial model. Are they verified here?

- Two outcomes (true he as either 5 is drawn or not),
- Finished number of trials (yes 10 draws here),

- Independent event (true here),
- Constant probability (true here).

Marks: 0, 3, 5

(ii) What is the probability that number 5 is one of the two lucky stars 3 times in the next ten draws? Give you result correct to 4 decimal places.

$$p(3) = C_{10}^{0}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{7} = 0.1550$$

Marks: 0, 3, 5

(iii) What is the probability that number 5 is one of the two lucky stars 2 times or less in the next ten draws? Give you result correct to 4 decimal places.

$$\begin{aligned} p(0) &= C_{10}^{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{10} = 0.1615 \\ p(1) &= C_{10}^{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{9} = 0.3230 \\ p(2) &= C_{10}^{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{8} = 0.2907 \\ p &= 0.165 + 0.3230 + 0.2907 = 0.7752 \end{aligned}$$

Marks: 0, 3, 5

(iv) What is the probability that number 5 is one of the two lucky stars 4 times or more in the next ten draws? Give you result correct to 4 decimal places.

$$p = 1 - 0.7752 - 0.155 = 0.0698$$

Marks: 0, 3, 5

Question 8 (50 Marks)

In a standard rail coach, there are 64 seats. The company makes a statistical study regarding the number of seats occupied. They find that at a given time of the day, the number of people in the coach follows a normal distribution with a mean of 45 people with a standard deviation of 4.

(a) What is the probability that there are less than 50 people in the coach?

$$p(X < 50) = p\left(Z < \frac{50 - 45}{4}\right) = p(Z < 1.25) = 0.8944$$

Marks: 0, 3, 5

(b) What is the probability that there are more than 38 people in the coach?

$$p(X > 38) = p\left(Z > \frac{38 - 45}{4}\right) = p(Z > -1.75) = p(Z < 1.75) = 0.9599$$

Marks: 0, 3, 5

(c) What is the probability that there are between 38 and 50 people in the coach?

$$p(38 < X < 50) = p(X < 50) - p(X < 38) = 0.8944 - (1 - .9599) = 0.8543$$

Marks: 0, 3, 5

(d) On average, the price paid by passenger in a coach between Limerick and Dublin are detailed below

Price paid	€14	€16	€20	€25	€40
Nb passenger	5	18	13	7	2

(i) Calculate the mean price paid by passengers.

$$\mu = 19.4$$

Marks: 0, 3, 5

(ii) Calculate the standard deviation of the price paid by passengers. Give your answer correct to 2 decimal places. Use the box on the next page if necessary.

$$\sigma = 5.63$$

Marks: 0, 3, 5

(e) The company carries out a study on occupancy rates of its coaches. 50 coaches are tested and on average, they are 60% full.

(i) Find a 95% confidence interval for the average occupancy rate of the coaches. Give your result as a percentage correct to two decimal places.

$$\left[.6 - 1.96\sqrt{\frac{0.6 \times 0.4}{50}} \quad , \quad .6 + 1.96\sqrt{\frac{0.6 \times 0.4}{50}} \right] \Longrightarrow [46.42, 73.58]$$

Marks: 0, 3, 5

(ii) The company believes that its coaches are 65% full. Test the company's claim using a 5% confidence level. Clearly state your null hypothesis, your alternative hypothesis, your intermediate calculations and your conclusions. Use the boxes on this page and on the next page for your answer.

- H_0=65%
- $H_a \neq 65\%$
- 65 is in the confidence interval so we do not reject H_0: at a 95% confidence level, coaches are 65% full.

Marks: 0, 3, 5

(f) The company believes than on average, passengers pay €23 for their ticket. They checked the price paid by 50 passengers. On average they each paid €21 with a standard deviation of €5.

(i) Test the company's claim using a 5% confidence level. Clearly state your null hypothesis, your alternative hypothesis, your intermediate calculations and your conclusions.

- P=23
- $P \neq 23$
- Statistic

$$Z = \frac{21-23}{\frac{5}{\sqrt{50}}} = -2.83$$

- The Z value is not in [-1.96, 1.96] so H_0 should be rejected: passenger do not pay €23 at a 95% confidence level.

Marks: 0, 3, 5, 7

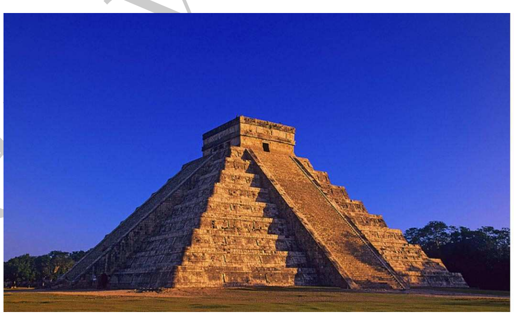
(ii) Find the p-value of the test you carried out in part (i) above and explain the meaning of this value.

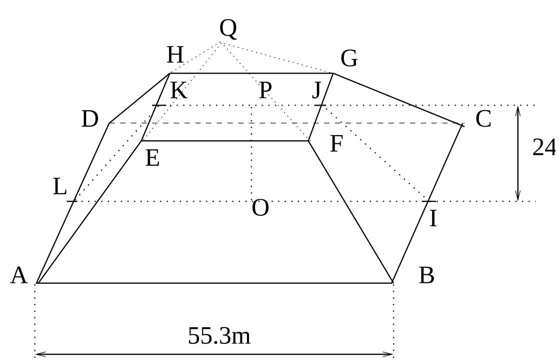
$$p_{value} = 2(1 - p(Z < 2.83)) = 2(1 - 0.9977) = 0.0046$$

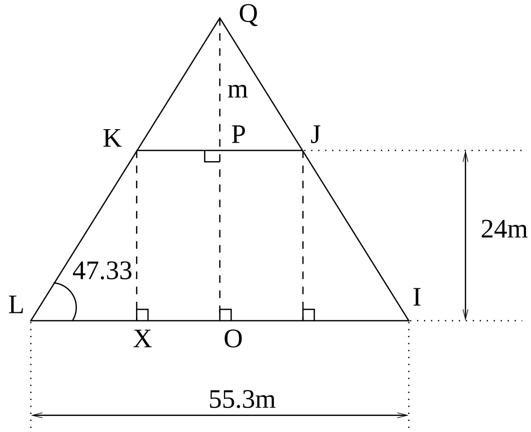
This is the maximum level of confidence that can be used to accept H_0.
Marks: 0, 3, 5, 8

Question 9 (50 Marks)

The temple of Kukulkan in Mexico, shown in the picture below, has the shape of a truncated pyramid with a square base of length 55.3m and a height of 24m. A cross section of the pyramid is IJKL. The (virtual) apex of the pyramid is Q, P is the midpoint of [JK] and the midpoint of [IL] is point O. ABCD and EFGH are squares. The slope angle is $< KLO = 47.33°$.







(a) Using the information given, calculate $|LX|$ and hence $|KP|$. Give your results correct to 3 decimal places. Use box on the next page if necessary.

$$|LX| = \frac{24}{\tan 47.33} = 22.123$$
$$|KP| = |OX| = |LO - |LX| = \frac{55.3}{2} - 22.123 = 5.527$$

Marks: 0, 3, 5

(b) Height of the virtual pyramid.

(i) Show that the two triangles KPQ and LOQ are similar.

$$< KQP =< LQO \quad < QPK =< QOL \Longrightarrow < OLQ =< PKQ$$

There are three equal angles so the triangles are similar.
Marks: 0, 3, 5

(ii) Hence, show the the distance $m = |PQ|$ verifies

$$\frac{m}{m+24} = \frac{5.527}{27.65}$$

$$\frac{|PQ|}{|OQ|} = \frac{|KP|}{|LO|} \Longleftrightarrow \frac{m}{m+24} = \frac{5.527}{27.65}$$

Marks: 0, 3, 5, 8, 10

(iii) Show that m=6 and therefore calculate the height of the virtual pyramid

$$\frac{m}{m+24} = \frac{5.527}{27.65} \quad \Longleftrightarrow \quad 27.65m = 5.527(m+24)$$
$$\Longleftrightarrow \quad (27.65 - 5.527) = 24 \times 5.527$$
$$\Longleftrightarrow \quad m = \frac{24 \times 5.527}{27.65 - 5.527} \approx 6$$

Marks: 0, 3, 5, 8, 10

(c) Volume of the temple.

(i) Calculate the volumes of the virtual pyramids ABCDQ and EFGHQ. Calculate your results correct to two decimal places.

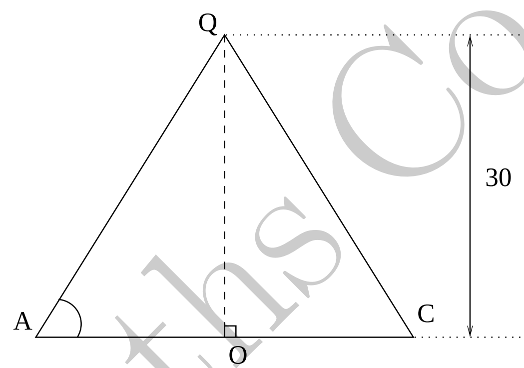
$$\begin{aligned} V_1 &= \frac{1}{3}(6)(2 \times 5.527)^2 = 244.38\ m^3 \\ V_2 &= \frac{1}{3}(30)(55.3)^2 = 30580.9\ m^3 \end{aligned}$$

Marks: 0, 3, 5

(ii) Hence calculate the volume of the truncated pyramid. Calculate your results correct to two decimal places.

$$V = 30580.9 - 244.38 = 30336.52\ m^3$$

(d) Angle of the egdes. O is the midpoint of [AC]



(i) Calculate $|AC|$ using the information provided at the start of the question. Give your answer correct to one decimal place.

$$|AC| = 55.3\sqrt{2} = 78.2$$

Marks: 0, 3, 5

(ii) Hence calculate angle $< QAC$. Calculate your result correct to two decimal places.

$$\tan < QAC = \frac{30}{\frac{78.2}{2}} = .7673 \Longrightarrow < QAC = 37.5^\circ$$

Marks: 0, 3, 5