Leaving Certificate Examination, 2018

Sample paper prepared by Leamy Maths Community

Mathematics

Paper 1

Higher Level

Solutions

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Name ____

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Total

300 marks

Section A	Concepts and Skills	150 Marks

Answer **all six** questions from this section.

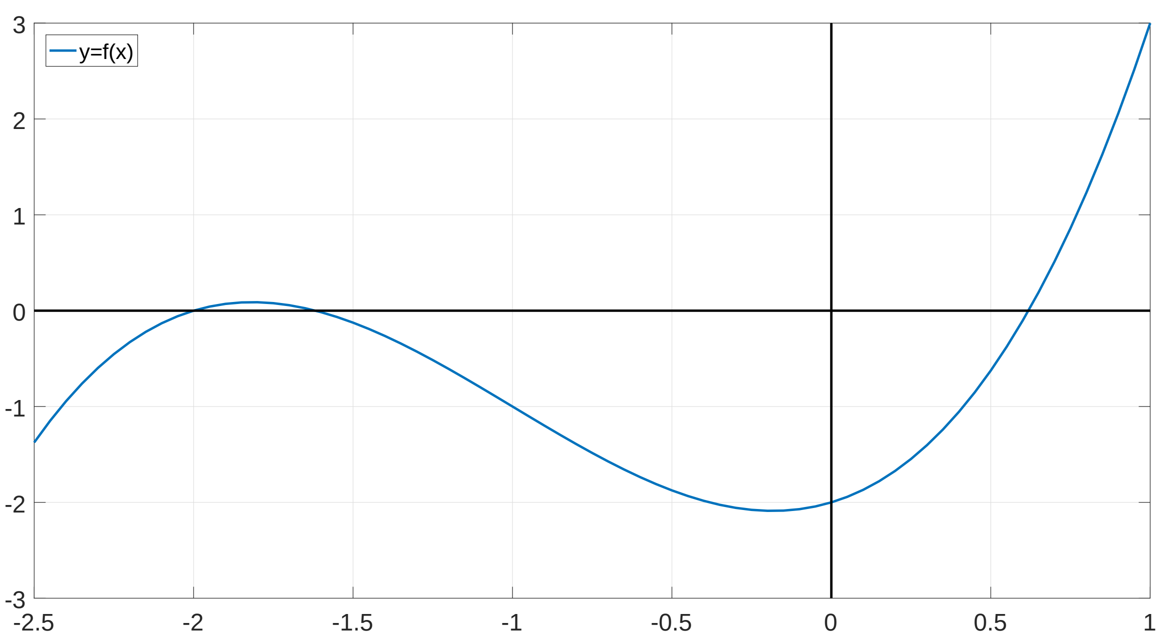
Question 1 (25 Marks)

(a) Using the graph below, identify a root x_0 of the function

$$f(x) = x^3 + 3x^2 + x - 2$$

Calculate $f(x_0)$ to check your answer.

$$f(-2) = -8 + 12 - 2 - 2 = 0$$



(b) Calculate the other solutions of the equation

$$x^3 + 3x^2 + x - 2 = 0$$

and express any irrational solution in the form $a + b\sqrt{c}$ where a, b and c are rational numbers.

$$f(x) = (x+2)\left(x^2 + x - 1\right)$$
$$x_1 = \frac{-1+\sqrt{5}}{2} \qquad x_2 = \frac{-1-\sqrt{5}}{2}$$

(c) Hence calculate the solutions of the equation

$$x^6 + 3x^4 + x^2 - 2 = 0$$

The solutions are the square roots of the positive solutions of equation $x^3 + 3x^2 + x - 2 = 0$.

$$x_1 = \sqrt{\frac{-1+\sqrt{5}}{2}} \qquad x_2 = -\sqrt{\frac{-1+\sqrt{5}}{2}}$$

Question 2 (25 Marks)

(a) Identify the turning points of function

$$f(x) = \frac{x^2}{x+1}$$

and specify if each tuning point is a minimum or a maximum

$$f'(x) = \frac{x^2+2x}{(x+1)^2} \qquad f''(x) = \frac{2}{(x+1)^3}$$
$$f'(x) = 0 \Longrightarrow x^2 + 2x = 0 \Longrightarrow x = 0 \quad x = -2$$
$$f''(0) = 2 > 0 \qquad f''(2) = -2 < 0$$

(0,0) is a minimum, (-2,-4) is a maximum

(b) What is the coefficient of the term independent of x in the expansion of

$$\left(2x - \frac{1}{x}\right)^8$$

$$^8C_4(2x)^4\left(-\frac{1}{x}\right)^4 = 1120$$

Question 3 (25 Marks)

The percentage of battery available in a tablet is given by the function:

$$f(t) = Ae^{Bt}$$

where t is in hours.

At t=0, the tablet is fully charged ($f(0) = 100\%$). After 3 hours, at t=3, the tablet is 40.66% charged ($f(3) = 40.66\%$).

(a) Calculate A and B correct to 3 decimal places.

$$A = 100$$
$$40.66 = 100e^{3B} \Longrightarrow B = \frac{1}{3}\ln(0.4066) = -0.3$$

(b) How much battery will be available at time t=5 hours? Give your results correct to 3 decimal places.

$$f(5) = 22.313\%$$

(c) Calculate when the tablet will only have 10% battery left. Give your results correct to 3 decimal places.

$$10 = 100e^{-0.3t} \Longrightarrow t = -\frac{1}{0.3}\ln(0.1) = 7.675hr$$

(d) Calculate the rate of change of the battery at time t=4 hours. Give your result correct to 3 decimal places.

$$f'(t) = -30e^{-0.3t} \Longrightarrow f'(5) = -9.036$$

Question 4 (25 Marks)

Gráinne wants to have a lump sum when she retires. She decides to save €250 at the beginning of every month for 30 years.

(a) The bank offers an AER of 6%. Show that this corresponds to a rate of 0.487% per month.

$$1 + r = (1 + 0.06)^{1/12} = 1.00487 \Longrightarrow r = 0.00487 = 0.487\%$$

(b) What is the value on the retirement date of a payment P made n months before the retirement?

$$Val = P(1.00487)^n$$

(c) What will be the value of the fund when Gráinne retires if she makes equal monthly payments of €250 for 30 years? Give you result correct to the nearest Euro.

$$Val = 250\left(1.00487^360 + 1.00487^359 + ... + 1.00487\right) = 250(1.00487)\frac{1.00487^360 - 1}{1.00487 - 1} = 244,952$$

(d) How much should she pay every month if she wants to have the same sum available when she retires but only wants to pay for 20 years? Give you result correct to the nearest Euro.

$$Val = P\left(1.00487^240 + 1.00487^239 + ... + 1.00487\right) = P(1.00487)\frac{1.00487^240 - 1}{1.00487 - 1} \Longrightarrow P = 537.4$$

Question 5 (25 Marks)

(a) Calculate the slope of the tangent at x=1 for the function

$$f(x) = \ln\left(1 + 2e^{x-1}\right)$$

$$f'(x) = \frac{2e^{x-1}}{1+2e^{x-1}} \implies f'(1) - \frac{2}{3}$$

(b) Differentiate the function

$$f(x) = 3x^2 + 4x - 5$$

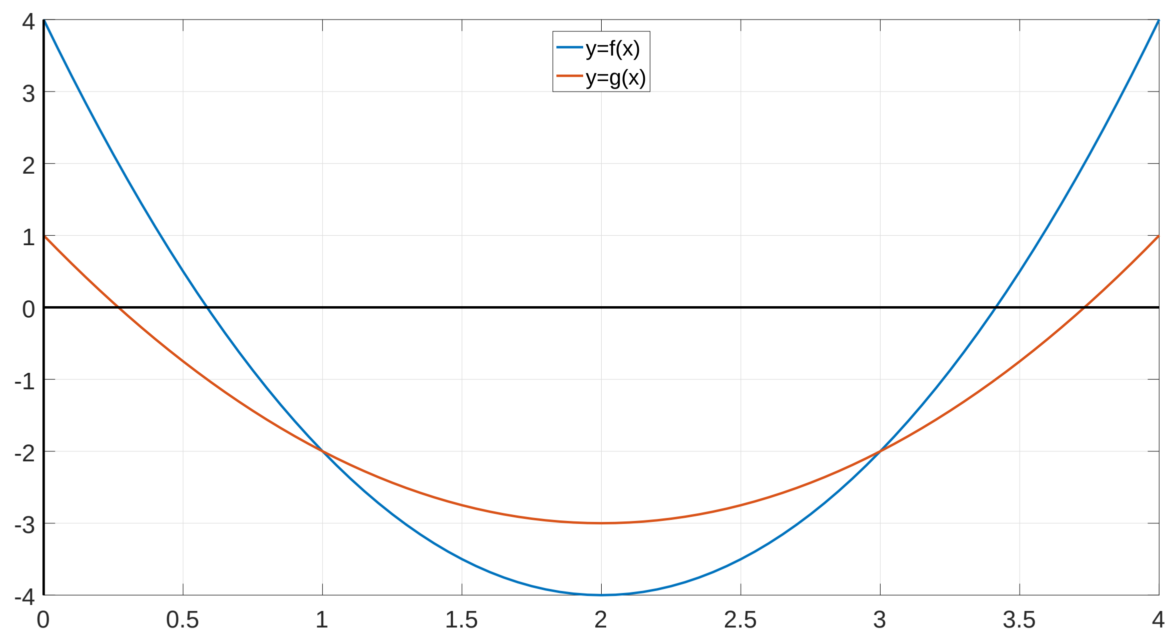
using first principles.

$$f(x+h) - f(x) = 6xh + h^2 + 4h \implies lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = 6x + 0 + 4 = 6x + 4$$

Question 6 (25 Marks)

(a) Plot the two functions $f(x)$ and $g(x)$ on the graph below.

$$f(x) = 2x^2 - 8x + 4 \qquad g(x) = x^2 - 4x + 1$$



(b) Identify where the two functions cross on the graph and verify this analytically.

$$2x^2 - 8x + 4 = x^2 - 4x + 1 \implies x^2 - 4x + 3 = 0 \implies x = 1 \quad x = 3$$

(c) Calculate the area between the two functions.

$$\int_1^3 (x^2 - 4x + 3)\, dx = \left[\frac{x^3}{3} + 2x^2 + 3x\right]_1^3 = (9 - 18 + 3) - \left(\frac{1}{3} - 2 + 1\right) = \frac{4}{3}$$

Section B	Contexts and Applications	150 Marks

Answer **all three** questions from this section.

Question 7 (50 Marks)

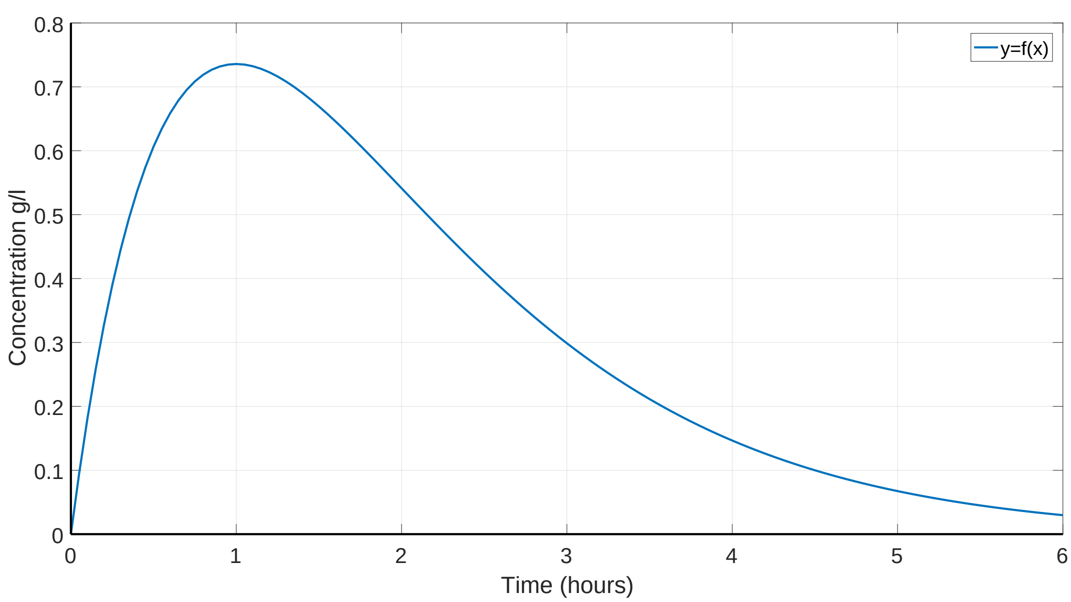
Paul has to ingest a drug on a regular basis. The concentration of drug in the blood after ingesting a pill is given by the function

$$C(t) = 2te^{-t}$$

where t represents the time in hours and C(t) is the drug concentration in g/l.

(a) Plot the curve on the graph below.

0	1	2	3	4	5	6
0	0.736	0.541	0.99	0.147	0.067	0.03



(b) To make sure the drug is most effective, its concentration in the blood should be above 0.5 g/l. Using the curve, identify for what values of t the drug is effective. Justify you answer and if necessary use the function plot.

You read on the curve that the concentration 0.5 g/l occurs for $0.357 < t < 2.153$

(c) As a precaution, it is recommended that Paul should not take another pill before the concentration of drug in blood reaches its maximum. Calculate the value of t when the concentration reaches its maximum. Calculate the value of the maximum and clearly identify this point on the curve.

$$C'(t) = 2(1-t)e^{-t}$$
$$C'(t) = 0 \Longrightarrow t - 1 = 0 \Longrightarrow t = 1$$

Maximum because C(0)=0¡C(1)= 0.736

(d) Hence identify for what values of t the concentration of the drug in the blood is increasing and decreasing.

The extremum occurs at $t = 1$. This is a maximum because C(0)=0¡C(1)= 0.736. The curve is increasing for $t < 1$ and decreasing for $t > 1$

(e) What happens to the function when $t \to \infty$? What does this mean for the curve and for the drug concentration?

- The curve is decreasing for t¿1.
- The function is always positive.
- The function will go down very close to 0 but always above 0.
- This is a horizontal asymptote
- This means that the concentration of drug in the body will become very small (however, in practice, the concentration will actually reach 0 in the blood).

(f) Paul has taken a pill at 10.00 and another pill at 12.00. What is the concentration of drug in Paul's blood at 13.00 and what will it be at 14.00?

- at t=13.00 $C = C(1) + C(3) = 0.299 + 0.736 = 1.034\ g/l$
- at t=14.00 $C = C(2) + C(4) = 0.147 + 0.541 = 0.688\ g/l$

Question 8 (50 Marks)

Origami or the art of paper folding has attracted a lot of attention from mathematicians. This sometimes leads to surprising results. Common belief is that you can not fold a piece of paper more than 7 or 8 times. In 2002, American high school student Britney Gallivan managed to fold a piece of paper 12 times, a record that was broken in 2012 by a group of students in Massachusetts who folded a piece of paper 13 times. In the following, you will investigate some of the mathematics associated with paper folding.

(a) Perfect paper folding.
You start with a square sheet of paper of size 1m×1m. This paper is 0.45mm thick. You fold it repetitively in alternate directions.

(i) Fill in the table below

No. of folds	0	1	2	3	4	5
No. of layers	1	2	4	8	16	32
Area (m^2)	1	0.5	0.25	0.125	0.0625	0.03125

(ii) Write a formula that describes the number of layers in terms of n, where n is the number times the sheet has been folded.

$$No\ of\ layers = 2^n$$

(iii) Show by induction that

$$a + a \cdot r + a \cdot r^2 + \ldots + a \cdot r^{n-1} = \frac{a\,(1-r^n)}{1-r}$$

(iv) A sheet of paper is 0.45mm thick. You lay down an unfolded sheet of paper, then add a sheet of paper which has been folded once on top of it, then put a sheet of paper which has been folded twice on top of it and so on. How many layers of paper are there in total if you repeat the process until you add a sheet of paper folded 10 times?

$$n = 2^0 + 2^1 + ... + 2^{10} = \frac{2^{11}-1}{2-1} = 2047$$

(v) If you continue the process described above, how many folded sheets of paper will you need to pile up to reach a height of 10m?

$$H_n = 0.45\left(2^0 + 2^1 + ... + 2^n\right) = 0.45\frac{2^{n+1}-1}{2-1} = 10,000$$
$$\Longrightarrow 2^{n+1} = \frac{10,000}{0.45} + 1$$
$$\Longrightarrow n = 14.43 - 1 = 13.43$$

The sheet needs to be folded 14 times so you need 15 sheets.

(b) Practical paper folding
It becomes more and more difficult to fold a sheet of paper once you have folded it a few times. The minimum length of a square paper sheet to be folded n times is given by:

$$W_n = \pi t 2^{3(n-1)/2}$$

where t is the thickness of the paper and n is the number of times the sheet of paper is folded

(i) If the paper has a thickness of 0.45mm, what is the length of paper required to match the record of folding the paper 13 times. Round up the result to the nearest metre.

$$W_{13} = 371m$$

(ii) According to the formula above, how many times can you fold a 0.1mm thick square sheet of paper 1.29m× 1.29m .

$$\begin{aligned} 1290 = \pi(0.1)2^{3(n-1)/2} &\implies 2^{3(n-1)/2} = 4106.2 \\ &\implies \frac{3(n-1)}{2} = \frac{ln(4106.2)}{ln(2)} = 12 \\ &\implies n = 9 \end{aligned}$$

Question 9 (50 Marks)

On 31 January 2018, there was a total lunar eclipse. A lunar eclipse occurs when the Moon passes directly behind the Earth into its umbra (shadow). This can occur only when the Sun, Earth, and Moon are aligned exactly, or very closely, with the Earth in the middle. Similarly, a solar eclipse (as seen from planet Earth) occurs when the Moon passes between the Sun and Earth, and when the Moon fully blocks (”occults”) the Sun. This can occur only when the Sun, Earth, and moon are aligned exactly, or very closely, with the Moon in the middle (information from Wikipedia). You can approximately determine potential dates of eclipses using complex numbers.

(a) Position of the Earth with respect to the Sun.
In this model, the Sun is at coordinates (0,0) and the position of the Earth is determined with the complex number

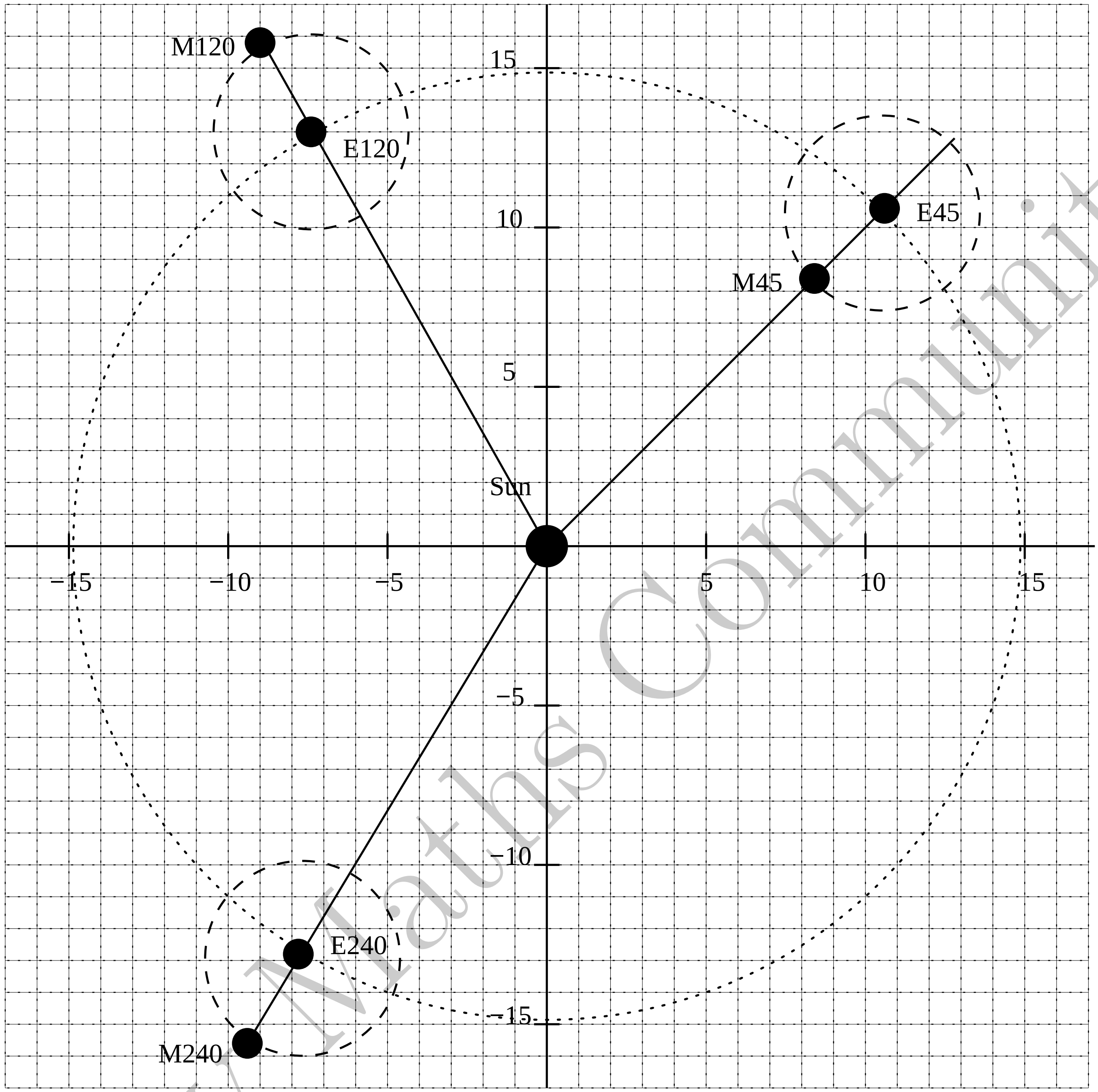
$$Z_E = 15\,(\cos t + i \sin t)$$

where t is the time in is in days. (Your calculator must be in **degrees**) and distances are scaled.

(i) Identify the modulus and the argument of Z_E.
Modulus=15 and argument=t

(ii) Calculate the position of the Earth at t=45 days, t=120 days and t=240 days and place them on the Argand diagram on the next page (calculate all results accurate to 2 decimal places). Identify the points clearly on the graph (Your calculator must be in **degrees**).

$$\begin{aligned} Z_{45} &= 10.6 + 10.6i \\ Z_{120} &= -7.5 + 13i \\ Z_{240} &= -7.5 - 13i \end{aligned}$$



(iii) Calculate the straight line distance (modulus) between the position of the Earth at t=45 days and the position of the Earth at t=240 days accurate to 2 decimal places.

$$|Z_{240} - Z_{45}| = |-18.1 - 23.6i| = \sqrt{18.1^2 + 23.^2} = 29.74$$

(b) Rotation of the Moon around the Earth
The position of the Moon with respect to the Sun is

$$Z_M = Z_E + Z_P$$

where

$$Z_P = 3\,(\cos 13t + i \sin 13t)$$

(Note the position is not up to scale but this does not affect results in the following).

(i) Identify the modulus and argument of Z_P
Modulus=3 and argument=13t

(ii) Calculate the position of the Moon at t=45 days, t=120 days and t=240 days and place them on the Argand diagram on the previous page (calculate all results accurate to 2 decimal places). Identify the points clearly on the graph. For which of these values of t are the Sun, Earth and Moon aligned in this order? (Your calculator must be in **degrees**).

$$\begin{aligned} M_{45} &= 8.47 + 8.47i \\ M_{120} &= -9 + 15.6i \\ M_{240} &= -9 - 15.6i \end{aligned}$$

Sun, Earth and Moon aligned in this order for $\theta = 120$ and $\theta = 240$

(iii) Show that

$$\frac{Z_M}{Z_E} = 1 + \frac{Z_P}{Z_E}$$

$$Z_M = Z_E + Z_P \Longrightarrow \frac{Z_M}{Z_E} = \frac{Z_E + Z_P}{Z_E} = \frac{Z_E}{Z_E} + \frac{Z_P}{Z_E} = 1 + \frac{Z_P}{Z_E}$$

(iv) You can show that when $\frac{Z_M}{Z_E}$ is a real number, then the Sun the Earth and the Moon are aligned. Calculate the modulus and argument of $\frac{Z_P}{Z_E}$ using the polar form and then using your answer for question (iii), express the fraction $\frac{Z_M}{Z_E}$ in the form

$$\frac{Z_M}{Z_E} = a + ib, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}$$

Modulus=3/15=1/5=0.2. Argument=13t-t=12t

$$a = 1 + \frac{1}{5}\cos(12t) \qquad b = \frac{1}{5}\sin(12t)$$

(v) Solve the equation

$$\sin 12t = 0$$

Give all answers for $0 \leq t \leq 180$. How often do you expect to see an eclipse according to the solutions.
t=0, 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180

(vi) Using the solutions of the previous question, calculate the calendar dates corresponding to the values of t you identified assuming t=0 corresponds to the 31 of January.
15 Feb, 2 Mar, 15 Mar, 1 Apr, 16 Apr, 1 May, 16 May, 3 May, 15 Jun, 30 Jun, 15 Jul and 30 Jul.

(vii) In 2018, eclipses or partial eclipses occur on 31 January, 15 February, 13 July, 28 July and 11 August. Did you get some of these dates (or close enough)? Why do you think eclipses do not occur for all the dates you identified?
The Earth and the Moon do not move in the same plane around the Sun so the Earth and the Moon do not always occult the Sun.