

Leaving Certificate Examination, 2023

Sample paper prepared by Leamy Maths Community

Mathematics

Paper 1

Higher Level

1 May 2023

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Solutions

300 marks

Sample Instructions

There are two sections in this examination paper:

Section A	Concepts and Skills	150 marks	5 of 6 questions
Section B	Contexts and Applications	150 marks	3 of 4 questions

Answer questions as follows:

In Section A, answer five questions.

In Section B, answer three questions.

Write your answers in the spaces provided in this booklet. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the booklet of *Formulae and Tables*. You must return it at the end of the examination.

You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Answer **all six** questions from this section.

Question 1

(25 Marks)

- (a) The equation $z^3 - z^2 - z - 2 = 0$ has one integer solution z_1 and two complex solutions $z_2 = a + ib$ and $z_3 = a - ib$ where $a, b \in \mathbb{R}$ and $b \geq 0$. Calculate the three solutions z_1, z_2 and z_3 of the equation.

$z_1 = 2$ is a solution as $2^3 - 2^2 - 2 - 2 = 8 - 4 - 2 - 2 = 0$ and therefore using long division $z^3 - z^2 - z - 2 = (z - 2)(z^2 + z + 1) = 0$. The solutions of the quadratic are

$$z_2 = \frac{-1 + i\sqrt{3}}{2} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_3 = \frac{-1 - i\sqrt{3}}{2} = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

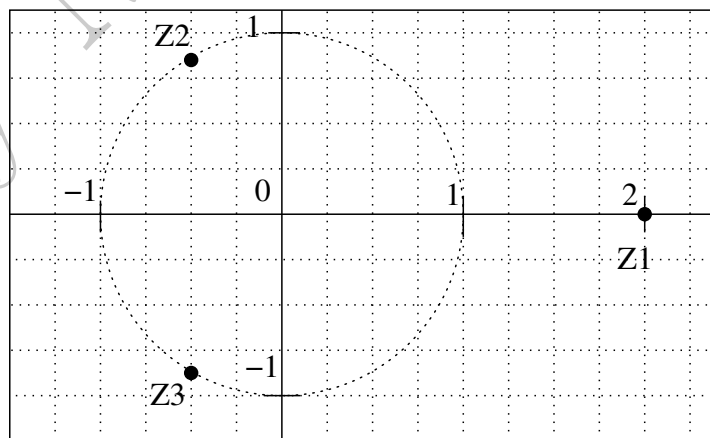
Marks: 0, 3, 5, 8, 10

- (b) Express the three solutions in polar form and plot them on the Argand diagram below.

$$z_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$z_3 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

Marks: 0, 3, 5, 8, 10



- (c) What unique geometrical transformation maps both z_2 to z_3 and z_3 to z_2 ?
Axial symmetry with respect to the x-axis.

Marks: 0, 3, 5

Question 2

(25 Marks)

(a) Solve the system

$$\begin{cases} a^2 + b^2 = 13 \\ ab = 6 \end{cases}$$

$$\begin{aligned} a = \frac{6}{b} &\implies \frac{36}{b^2} + b^2 = 13 \\ &\implies b^4 - 13b^2 + 36 = 0 \\ &\implies b^2 = 4 \quad \text{or} \quad b^2 = 9 \\ &\implies b = \pm 2 \quad \text{or} \quad b = \pm 3 \end{aligned}$$

Solutions (2,3), (-2,-3), (3,2), (-3,-2)

Marks: 0, 3, 5, 8, 10

(b) Solve the inequality

$$\frac{2x-3}{x+1} > 1$$

$$\begin{aligned} \frac{2x-3}{x+1} > 1 &\iff (2x-3)(x+1) > (x+1)^2 \\ &\iff 2x^2 - 3x + 2x - 3 > x^2 + 2x + 1 \\ &\iff x^2 - 3x - 4 > 0 \end{aligned}$$

$x < -1$ or $x > 4$

Marks: 0, 3, 5, 8, 10

(c) Solve the inequality

$$\left(\frac{1}{2}\right)^x > 2$$

$$\begin{aligned} \left(\frac{1}{2}\right)^x > 2 &\iff x \ln \left(\frac{1}{2}\right) > \ln 2 \\ &\iff -x \ln 2 > \ln 2 \\ &\iff x < -1 \end{aligned}$$

Marks: 0, 3, 5

Question 3

(25 Marks)

(a) Differentiate the function $f(x) = x^2 - 3x + 8$ from first principles.

$$\begin{aligned} f(x+h) &= (x+h)^2 - 3(x+h) + 8 = x^2 + 2xh + h^2 - 3x - 3h + 8 \\ &\iff \frac{f(x+h) - f(x)}{h} = 2x + h - 3 \\ &\iff f'(x) = 2x - 3 \end{aligned}$$

Marks: 0, 3, 5, 8, 10

- (b) Calculate the equation of the tangent for function $g(x)$ at point $x = -1$. Use space on the next page if necessary.

$$g(x) = xe^{x^2+x}$$

$$g'(x) = e^{x^2+x} + x(2x+1)e^{x^2+x} = e^{x^2+x}(2x^2+x+1)$$

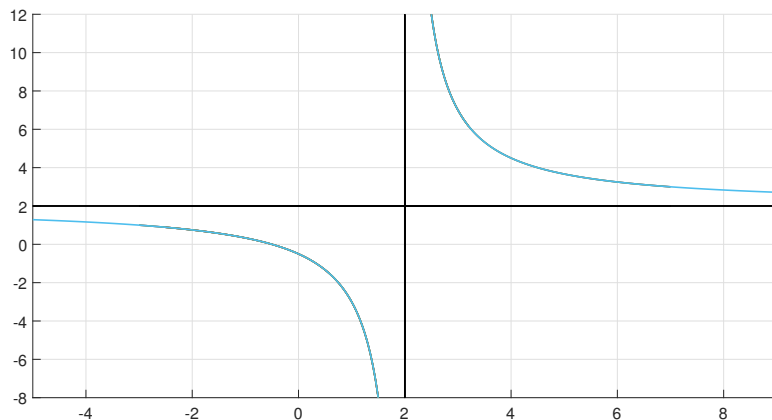
$g(-1) = -1$, $g'(-1) = 2$ so the tangent is $y + 1 = 2(x + 1)$ or $y = 2x + 1$.

Marks: 0, 3, 5, 8, 10

- (c) Identify the asymptotes of function and specify if they are vertical or horizontal asymptotes. Plot the asymptotes and roughly sketch function $h(x)$ using the grid on the next page.

$$h(x) = \frac{2x+1}{x-2}$$

Horizontal: $y = 2$ Vertical: $x = 2$ Marks: 0, 3, 5



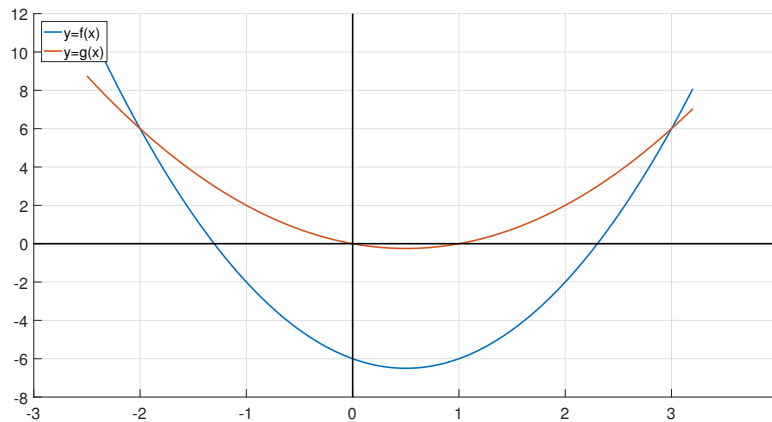
Question 4

(25 Marks)

- (a) Plot the two functions $f(x)$ and $g(x)$. Use the grid on the next page and the space below for your rough work.

$$f(x) = 2x^2 - 2x - 6 \quad g(x) = x^2 - x$$

Marks: 0, 3, 5



- (b) Identify where the two function cross on the graph and verify this analytically.

$$\begin{aligned} f(x) = g(x) &\iff 2x^2 - 2x - 6 = x^2 - x \\ &\iff x^2 - x - 6 = 0 \\ &\iff (x + 2)(x - 3) = 0 \end{aligned}$$

(-2,6) and (3,6)

Marks: 0, 3, 5, 8

- (c) Calculate the area between the two functions. Give your result in the form $\frac{a}{b}$ where $a, b \in \mathbb{N}$.

$$\begin{aligned} \int_{-2}^3 (-x^2 + x + 6) dx &= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x \right]_{-2}^3 \\ &= \left[-9 + \frac{9}{2} + 18 \right] - \left[-\frac{8}{3} + 2 - 12 \right] \\ &= \frac{125}{6} \end{aligned}$$

Marks: 0, 3, 5, 8, 10, 12

Question 5

(25 Marks)

The first three terms of a series are: $1, x + 2, x^2 + 3$

(a) Arithmetic series

(i) Find the values of x if the series is arithmetic.

$$x + 2 - 1 = x^2 + 3 - (x + 2) \iff x^2 - 2x = 0$$

$$x = 0 \text{ and } x = 2$$

Marks: 0, 3, 5, 8

(ii) Hence, for both values of x from part (i), write the general term of the series in the form $T_n = p + nq$ where $p, q \in \mathbb{Z}$.

$$x = 0: T_n = n \text{ and } x = 2: T_n = 3n - 2$$

Marks: 0, 3, 5

(b) Geometric series

(i) Find the values of x if the series is geometric.

$$\frac{x+2}{1} = \frac{x^2+3}{x+2} \iff x^2 + 4x + 4 = x^2 + 3$$
$$\iff x = -\frac{1}{4}$$

Marks: 0, 3, 5, 7

(ii) Calculate the value of the 4th term of this series. Give your result in the form $\frac{a}{b}$ where $a, b \in \mathbb{N}$

$$T_1 = 1 \quad T_2 = \frac{7}{4} \quad T_3 = \frac{49}{16} \quad T_4 = \frac{343}{64}$$

Marks: 0, 3, 5

Question 6

(25 Marks)

In an experiment, meteorologists are observing the formation of rain drops. The drops are spherical and their radius increases at the rate of $2 \mu\text{m/s}$ where μm is a small fraction of a metre.

(a) Find the rate of increase of the droplet volume when the radius is $5 \mu\text{m}$. Give your result in the form $a\pi$ where $a \in \mathbb{N}$.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(5)^2(2) = 200\pi$$

Marks: 0, 3, 5

- (b) Calculate the rate of increase for the surface of the sphere when the rate of increase for the volume of the sphere is $288\pi (\mu m)^2$.

$$\frac{dV}{dt} = 288\pi = \pi r^2(2) \iff r^2 = \frac{288\pi}{8\pi} = 36 \implies r = 6$$

$$A = 4\pi r^2 \iff \frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(6)(2) = 96\pi$$

Marks: 0, 3, 5, 8, 10, 12

- (c) The droplet stops being spherical when the rate of increase of the surface area grows at a pace larger than the critical value $128\pi (\mu m)^2/s$. Calculate the radius of the droplet when the surface increases at this critical rate.

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r(2) = 16\pi r = 128\pi \iff r = \frac{128\pi}{16\pi} = 8$$

Marks: 0, 3, 5

- (d) Is it possible to obtain spherical droplets with a radius $r=10\mu m$? Justify your answer.
No as this is larger than the critical radius.

Marks: 0, 3

Answer **all three** questions from this section.

Question 7

(50 Marks)

An engineering company has to build a new concrete dam. Concrete is a mixture of water, cement and sand or gravel. When these ingredients are mixed, a chemical reaction occurs which creates a lot of heat. To prevent this heat from damaging the structure of the dam, a network of pipes is installed inside the dam to cool it down.

The average temperature in the dam is calculated using Newton's law of cooling:

$$T = T_a + (T_0 - T_a)e^{-kt}$$

where t is the **time in months**, T_a is the ambient or surrounding temperature, T_0 is the temperature at time $t = 0$ and k is a positive constant.

- (a) Engineers want to estimate how long it will take the dam to cool down without water flowing through the pipes. They think that a couple of months after completion, the average temperature in the dam will be $T_0 = 60^\circ\text{C}$ (this corresponds to $t=0$). They estimate that two months later, the average temperature in the dam will be $T = 59.75^\circ\text{C}$.
- (i) Using these two temperatures estimated by the engineers and using an average ambient temperature of 10°C , identify T_a and calculate k correct to 4 decimal places.

$$\begin{aligned} T &= 10 + 50e^{-kt} \iff 59.75 = 10 + 50e^{-2k} \\ &\iff k = -\frac{1}{2} \ln \left(\frac{49.75}{50} \right) = 0.0025 \end{aligned}$$

Marks: 0, 3, 5, 8, 10

- (ii) What will be the average temperature in the dam after 50 years? Give your results correct to 1 decimal place.

$$T = 10 + 50e^{-0.0025(600)} = 21.2^\circ$$

Marks: 0, 3, 5

- (iii) How long will it take the dam to reach 12°C (express your answer in years)? Does this make sense?

$$12 = 10 + 50e^{-0.0025t} \iff t = -\frac{1}{0.0025} \ln \left(\frac{2}{50} \right) = 1287$$

This is 1287 months or 107 years and 3 months: way too long.

Marks: 0, 3, 5, 7

- (b) To reduce the cooling down time, engineers decide that the dam will not be built as a single concrete block and that cold water will be flown through the pipe system. In this situation, the average temperature in the dam is

$$T = 10 + 50e^{-0.0025t(2t-a)}$$

- (i) Calculate the value of a for which, at $t=2$: $10 + 50e^{-0.0025t(2t-a)} = 10 + 50e^{-0.0025t}$

$$\begin{aligned} 10 + 50e^{-0.0025t(2t-a)} &= 10 + 50e^{-0.0025t} \implies e^{-0.0025t(2t-a)} = e^{-0.0025t} \\ &\implies -0.0025t(2t-a) = -0.0025t \\ &\implies 2t-a = 1 \\ &\implies a = 2t-1 = 2(2)-1 = 3 \end{aligned}$$

Marks: 0, 3, 5, 8

- (ii) For what value of the time t is the following inequality true?

$$10 + 50e^{-0.0025t(2t-3)} > 10 + 50e^{-0.0025t}$$

$$\begin{aligned} 10 + 50e^{-0.0025t(2t-3)} > 10 + 50e^{-0.0025t} &\implies e^{-0.0025t(2t-3)} > e^{-0.0025t} \\ &\implies -0.0025t(2t-3) > -0.0025t \\ &\implies 2t-3 < 1 \\ &\implies t < 2 \end{aligned}$$

Marks: 0, 3, 5, 8, 10

- (iii) Calculate the average temperature in the dam after 2 years, correct to one decimal place when the new construction method is used.

$$T = 10 + 50e^{-0.0025(24)(2(24)-3)} = 13.4^\circ$$

Marks: 0, 3, 5

- (iv) Interpret the results of the two previous questions for the dam.

More realistic

Marks: 0, 3, 5

Question 8

(50 Marks)

Munster wants to build a second stadium in Limerick. To finance this project, they decide to issue a 20-year bond.

- (a) If the bond offers a 50% return on investment, what is the AER of this bond? Give your result correct to 3 decimal places.

$$1.5 = (1+r)^{20} \iff r = 0.02048$$

$$r=2.048\%$$

Marks: 0, 3, 5

(b) The bond will actually pay €6 at the end of every year plus an additional €100 lump sum at the end of the 20 years.

- (i) Calculate the present value of the €100 lump sum if the expected market interest rate is 4%. Give your result correct to 4 decimal places.

$$P = \frac{100}{1.04^{20}} = 45.6387$$

Marks: 0, 3

- (ii) Prove by induction that

$$a + ar + ar^2 + \dots ar^{n-1} = a \frac{1 - r^n}{1 - r}$$

proof:

- True for $n=1$

$$a \frac{1 - r^1}{1 - r} = a$$

- Assume true at level n

$$a + ar + ar^2 + \dots ar^{n-1} = a \frac{1 - r^n}{1 - r}$$

Then

$$\begin{aligned} a + ar + ar^2 + \dots ar^{n-1} + ar^n &= a \frac{1 - r^n}{1 - r} + ar^n \\ &= a \frac{1 - r^n + r^n(1 - r)}{1 - r} \\ &= a \frac{1 - r^n + r^n - r^{n+1}}{1 - r} \\ &= a \frac{1 - r^{n+1}}{1 - r} \end{aligned}$$

This is the property at level $n+1$ so the property is inductive

- The property is inductive, true for $n=1$ for true for all $n \in \mathbb{N}$

Marks: 0, 3, 5, 8, 10

- (iii) Hence show that if the expected market interest rate is 4%, the present value of all coupon payments is equal to

$$V = 6 \frac{1 - \frac{1}{1.04^{20}}}{0.04}$$

$$\begin{aligned} V &= \frac{6}{1.04} + \frac{6}{1.04^2} + \dots + \frac{6}{1.04^{20}} \Rightarrow a = \frac{6}{1.04} r = \frac{1}{1.04} \\ &\Rightarrow V = \frac{6}{1.04} \frac{1 - \frac{1}{1.04^{20}}}{1 - \frac{1}{1.04}} = 6 \frac{1 - \frac{1}{1.04^{20}}}{0.04} \end{aligned}$$

Marks: 0, 3, 5, 7

- (iv) Calculate the total present value of bond if the expected market interest rate is 4%. Give your result correct to 4 decimal places.

$$V + P = 81.5420 + 45.6387 = 127.1807$$

Marks: 0, 3, 5

- (v) The bond is sold for €120. Should you buy it? Justify your answer.
Buy, worth more than its sell value Marks: 0, 3

- (c) The income generated by ticket sales corresponds to €N per bond at the end of each year, so management can set aside €N-6 at the end of every year.

- (i) Show that after 20 years, if the expected rate is 4%, management has set aside

$$M = (N - 6) \frac{1.04^{20} - 1}{0.04}$$

$$\begin{aligned} M &= (N - 6) + (N - 6)(1.04) + (N - 6)1.04^2 + \dots + (N - 6)1.04^{19} \\ &= (N - 6) \frac{1.04^{20} - 1}{1.04 - 1} \\ &= (N - 6) \frac{1.04^{20} - 1}{0.04} \end{aligned}$$

Marks: 0, 3, 5, 7

- (ii) If every year ticket sales generate €12 per bond. How much money can management put aside in 20 years? Give your result correct to 2 decimal places.

$$M = 178.67$$

Marks: 0, 3, 5

- (iii) How much should ticket sales generate every year so management can exactly repay the final €100 of the bond after 20 years? Give your answer correct to 3 decimal places.

$$\begin{aligned} 100 &= N - 6 \frac{1.04^{20} - 1}{0.04} \implies N - 6 = 100 \frac{0.04}{1.04^{20} - 1} = 3.358 \\ &\implies N = 9.358 \end{aligned}$$

Marks: 0, 3, 5

Question 9

(50 Marks)

Consider the function

$$f(x) = \ln(1 + x^2)$$

- (a) Function properties

- (i) What are the domain and range of the function?
Domain is $\mathbb{R}$ and Range = $[0, +\infty)$

Marks: 0, 3, 5

- (ii) Show that $f(x) \geq 0$ for all $x \in \mathbb{R}$. Is the function surjective? Justify your answer.

$$x^2 \geq 0 \iff x^2 + 1 \geq 1 \iff \ln(x^2 + 1) \geq \ln 1 \iff \ln(x^2 + 1) \geq 0 \iff f(x) \geq 0$$

No x value for $y = -1$ so the function is not surjective. Marks: 0, 3, 5

- (iii) Show that $f(2) = f(-2)$. Is the function injective? Justify your answer.

$$f(2) = f(-2) = \ln 5$$

Two values of x for $y = \ln 5$ so the function is not injective. Marks: 0, 3, 5

- (iv) Calculate the inverse of the function for the limited domain $x \geq 0$.

$$\begin{aligned} y = \ln(x^2 + 1) &\iff e^y = x^2 + 1 \\ &\iff x = \sqrt{e^y - 1} \\ &\iff f^{-1}(x) = \sqrt{e^x - 1} \end{aligned}$$

Marks: 0, 3, 5, 7

- (b) Some calculus

- (i) Calculate the first order derivative of the function.

$$f'(x) = \frac{2x}{1 + x^2}$$

Marks: 0, 3, 5

- (ii) Calculate the second order derivative of the function.

$$f''(x) = \frac{2 - 2x^2}{(1 + x^2)^2}$$

Marks: 0, 3, 5

- (iii) Identify the turning point(s) and specify if each turning point is a minimum or a maximum.

$f'(x) = 0$ so $x = 0$ is the only turning point. $f''(0) = 2$ so $(0, 0)$ is a minimum.

Marks: 0, 3, 5

- (iv) Calculate the coordinates of the inflection point(s).

$f''(x) = 0 \implies 2 - 2x^2 = 0$ so the inflection points are $(1, \ln 2)$ and $(-1, -\ln 2)$

Marks: 0, 3, 5

- (v) Using one of the previous results, calculate

$$\int_0^1 \frac{2x}{1 + x^2} dx$$

$$\int_0^1 \frac{2x}{1 + x^2} dx = f(1) - f(0) = \ln 2$$

Marks: 0, 3, 5, 8

Question 10

(50 Marks)

On 31 January 2018, there was a total lunar eclipse. A lunar eclipse occurs when the Moon passes directly behind the Earth into its umbra (shadow). This can occur only when the Sun, Earth, and Moon are aligned exactly, or very closely, with the Earth in the middle. Similarly, a solar eclipse (as seen from planet Earth) occurs when the Moon passes between the Sun and Earth, and when the Moon fully blocks ("occults") the Sun. This can occur only when the Sun, Earth, and moon are aligned exactly, or very closely, with the Moon in the middle (information from Wikipedia). You can approximately determine potential dates of eclipses using complex numbers.

- (a) Position of the Earth with respect to the Sun.

In this model, the Sun is at coordinates (0,0) and the position of the Earth is determined with the complex number

$$Z_E = 15(\cos t + i \sin t)$$

where t is the time in is in days. (Your calculator must be in **degrees**) and distances are scaled.

- (i) Identify the modulus and the argument of Z_E .

Modulus=15 and argument= t

Marks: 0, 3

- (ii) Calculate the position of the Earth at $t=45$ days, $t=120$ days and $t=240$ days and place them on the Argand diagram on the next page (calculate all results accurate to 2 decimal places). Identify the points clearly on the graph (Your calculator must be in **degrees**).

$$Z_{45} = 10.6 + 10.6i$$

$$Z_{120} = -7.5 + 13i$$

$$Z_{240} = -7.5 - 13i$$

Marks: 0, 3, 5

- (iii) Calculate the straight line distance (modulus) between the position of the Earth at $t=45$ days and the position of the Earth at $t=240$ days accurate to 2 decimal places.

$$|Z_{240} - Z_{45}| = |-18.1 - 23.6i| = \sqrt{18.1^2 + 23.6^2} = 29.74$$

Marks: 0, 3, 5

- (b) Rotation of the Moon around the Earth

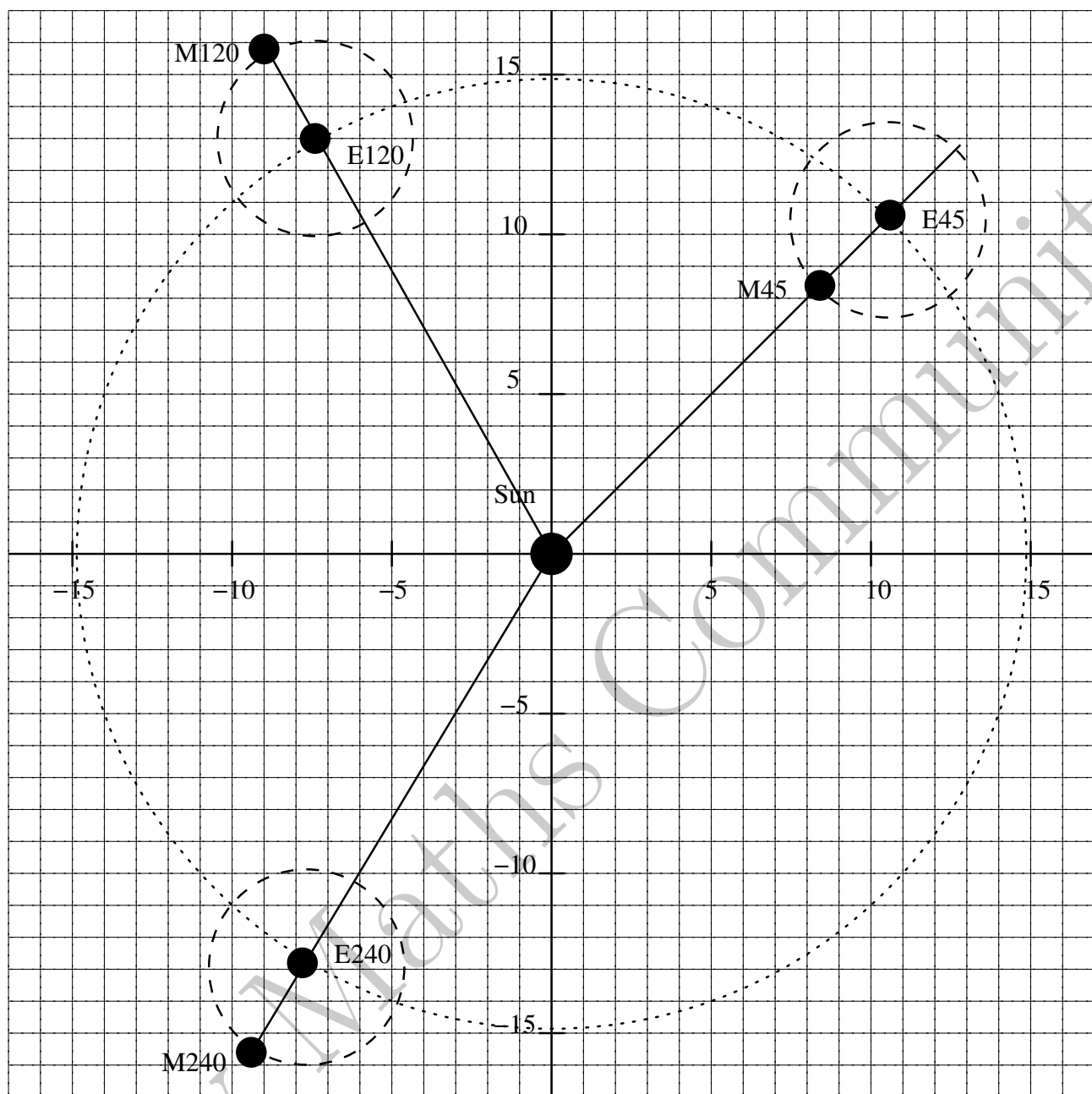
The position of the Moon with respect to the Sun is

$$Z_M = Z_E + Z_P$$

where

$$Z_P = 3(\cos 13t + i \sin 13t)$$

(Note the position is not up to scale but this does not affect results in the following).



- (i) Identify the modulus and argument of Z_P
Modulus=3 and argument= $13t$

Marks: 0, 2

- (ii) Calculate the position of the Moon at $t=45$ days, $t=120$ days and $t=240$ days and place them on the Argand diagram on the previous page (calculate all results accurate to 2 decimal places). Identify the points clearly on the graph. For which of these values of t are the Sun, Earth and Moon aligned in this order? (Your calculator must be in

degrees).

$$M_{45} = 8.47 + 8.47i$$

$$M_{120} = -9 + 15.6i$$

$$M_{240} = -9 - 15.6i$$

Sun, Earth and Moon aligned in this order for $\theta = 120$ and $\theta = 240$

Marks: 0, 3, 5

(iii) Show that

$$\frac{Z_M}{Z_E} = 1 + \frac{Z_P}{Z_E}$$

$$Z_M = Z_E + Z_P \implies \frac{Z_M}{Z_E} = \frac{Z_E + Z_P}{Z_E} = \frac{Z_E}{Z_E} + \frac{Z_P}{Z_E} = 1 + \frac{Z_P}{Z_E}$$

Marks: 0, 3, 5

(iv) You can show that when $\frac{Z_M}{Z_E}$ is a real number, then the Sun the Earth and the Moon are aligned. Calculate the modulus and argument of $\frac{Z_P}{Z_E}$ using the polar form and then using your answer for question (iii), express the fraction $\frac{Z_M}{Z_E}$ in the form

$$\frac{Z_M}{Z_E} = a + ib, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}$$

Modulus=3/15=1/5=0.2. Argument=13t-t=12t

$$a = 1 + \frac{1}{5} \cos(12t) \quad b = \frac{1}{5} \sin(12t)$$

Marks: 0, 3, 5

(v) Solve the equation

$$\sin 12t = 0$$

Give all answers for $0 \leq t \leq 180$. How often do you expect to see an eclipse according to the solutions.

t=0, 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180

Marks: 0, 3, 5, 8, 10

(vi) Using the solutions of the previous question, calculate the calendar dates corresponding to the values of t you identified assuming t=0 corresponds to the 31 of January.

15 Feb, 2 Mar, 15 Mar, 1 Apr, 16 Apr, 1 May, 16 May, 3 May, 15 Jun, 30 Jun, 15 Jul and 30 Jul.

Marks: 0, 3, 5

(vii) In 2018, eclipses or partial eclipses occur on 31 January, 15 February, 13 July, 28 July and 11 August. Did you get some of these dates (or close enough)? Why do you think eclipses do not occur for all the dates you identified?

The Earth and the Moon do not move in the same plane around the Sun so the Earth and the Moon do not always occult the Sun.

Marks: 0, 3, 5